StudForsk Project Report

Project name: United representation rings of finite groups
Student: Johanne Haugland
Supervisor: Markus Szymik

Symmetries are everywhere, and mathematicians study them using the algebraic theory of groups. Abstract mathematical groups, in turn, can be studied by representing their elements as linear transformations on complex vector spaces in a way that respects multiplication. This is the basic idea of representation theory.

The representations of a finite group G can be organized into the representation ring $\mathrm{R}(G)$ of that group. Each representation has a character that is just the function $G \to \mathbb{C}$ that associates to a group element the trace of the corresponding matrix. In this way the representation ring and the character ring of a group are isomorphic, and we can interchange them as we please. Typically, working with the character ring is better for calculations, whereas the representation ring fits better into conceptual frameworks.

The representation ring of a group is a weak invariant of that group: There are many non-isomorphic groups that have isomorphic representation rings. In a sense the smallest example is given by the dihedral group

$$\mathrm{D} = \langle s, d \mid s^2, (sd)^2, d^4 \rangle$$

of order 8 and the quaternion group

$$\mathrm{Q} = \{\pm 1, \pm \mathrm{i}, \pm \mathrm{j}, \pm \mathrm{k}\}$$

of the same order. The character tables

D	e	d^2	d	s	sd
Q	1	-1	i	j	k
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	-1	1
χ_4	1	1	-1	1	-1
χ_5	2	-2	0	0	0

of the two groups are identical up to renaming of the conjugacy classes. Hence, the character rings of the two groups are isomorphic. Both rings are isomorphic to the ring

$$(\star) \qquad \mathbb{Z}[\,x, y, z\,]/(\,x^2 = y^2 = 1, xz = yz = z, z^2 = 1 + x + y + xy\,),$$

as a direct inspection shows.

This situation prompts the problem of finding refinements of the representation ring that give stronger invariants of groups. We will first describe one standard way to do this, using Adams operations, and then propose a new approach: united representation rings. Finally, we will briefly discuss how to understand the relation between the two approaches.

Adams Operations

Let G be a finite group, and k be an integer. The function

$$\Psi^k\colon \mathrm{R}(G) \to \mathrm{R}(G)$$
$$\chi \mapsto \Psi^k\chi$$

with $(\Psi^k\chi)(g) = \chi(g^k)$ for all g in G is the k-th Adams operation on $\mathrm{R}(G)$. It is not immediate (but true) that Ψ^k lies in $\mathrm{R}(G)$ again, but it clearly is a ring homomorphism. For instance, we see that $\Psi^1 = \mathrm{id}$ is the identity map, and $\Psi^0(\chi) = \dim(\chi)$ is given by the dimension. Less obviously, the ring homomorphism Ψ^{-1} is conjugation.

We are mostly interested in the second Adams operation Ψ^2. On groups of odd order the map $g \mapsto g^2$ is a bijection, and consequently Ψ^2 is easy to describe on these groups: It permutes the irreducible representation, and it acts on a general character by transforming the values according to a Galois automorphism. For that reason, one cannot expect to learn much about such groups from Ψ^2, and we turn our attention towards the other extreme: 2-groups, that is groups whose order is a power of 2.

An Example

We now want to show that one can distinguish D and Q by looking at their representation rings together with Ψ^0 and Ψ^2. In other words, we want to show that there exists no ring isomorphism that commutes with both Ψ^0 and Ψ^2. To prove it, we assume that there exists a ring isomorphism $f\colon \mathrm{R}(\mathrm{Q}) \to \mathrm{R}(\mathrm{D})$ as described above, and will show that this leads to a contradiction.

From the character table we can see that all the one-dimensional characters square to 1. It is easy to check that there are only eight elements in the ring ($\star$) that square to 1, namely the obvious ones: ± 1, $\pm x$, $\pm y$, and $\pm xy$.

We assume that f commutes with Ψ^0, and this implies that f sends characters in R(Q) to characters in R(D) of the same dimension. As a consequence of this and $f(1) = 1$, the homomorphism f induces a bijection

$$\{x_{\mathrm{Q}}, y_{\mathrm{Q}}, x_{\mathrm{Q}} y_{\mathrm{Q}}\} \cong \{x_{\mathrm{D}}, y_{\mathrm{D}}, x_{\mathrm{D}} y_{\mathrm{D}}\},$$

where here and in the following we denote the set of irreducible characters in R(D) by $\{1, x_{\mathrm{D}}, y_{\mathrm{D}}, x_{\mathrm{D}} y_{\mathrm{D}}, z_{\mathrm{D}}\}$, and similarly for Q.

We will now look at consequences of the commutativity of f and Ψ^2, in particular of its implication $f\Psi^2(z_{\mathrm{Q}}) = \Psi^2 f(z_{\mathrm{Q}})$.

On the one hand, we have

$$\Psi^2(z_{\mathrm{Q}}) = -1 + x_{\mathrm{Q}} + y_{\mathrm{Q}} + x_{\mathrm{Q}} y_{\mathrm{Q}},$$

so that the left hand side can be rewritten as

$$\begin{aligned} f\Psi^2(z_{\mathrm{Q}}) &= f(-1 + x_{\mathrm{Q}} + y_{\mathrm{Q}} + x_{\mathrm{Q}} y_{\mathrm{Q}}) \\ &= -f(1) + f(x_{\mathrm{Q}}) + f(y_{\mathrm{Q}}) + f(x_{\mathrm{Q}} y_{\mathrm{Q}}). \end{aligned}$$

Using the action on the 1-dimensional characters established above, we end up with

$$f\Psi^2(z_{\mathrm{Q}}) = -1 + x_{\mathrm{D}} + y_{\mathrm{D}} + x_{\mathrm{D}} y_{\mathrm{D}}.$$

On the other hand, we can certainly express $\Psi^2 f(z_{\mathrm{Q}})$ as a linear combination of the irreducible representations in R(D) with integer coefficients, say

$$\Psi^2 f(z_{\mathrm{Q}}) = a1 + bx_{\mathrm{D}} + cy_{\mathrm{D}} + dx_{\mathrm{D}} y_{\mathrm{D}} + ez_{\mathrm{D}}.$$

Since f commutes with Ψ^0, we have $(f(z_{\mathrm{Q}}))(e) = 2$. Using that d is the only element in D that does not square to e, we see that

$$\Psi^2(f(z_{\mathrm{Q}}))(g) = 2 \text{ for } g \neq d.$$

Evaluating this expression for each element in D, we get the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & -2 \\ 1 & 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ (\Psi^2 f(z_{\mathrm{Q}}))(d) \\ 2 \\ 2 \end{bmatrix}.$$

Solving this system of linear equations allows us to deduce

$$\Psi^2 f(z_{\mathrm{Q}}) = (b+2)1 + bx_{\mathrm{D}} - by_{\mathrm{D}} - bx_{\mathrm{D}}y_{\mathrm{D}}.$$

Combining what we have learnt about the coefficients of $f\Psi^2(z_{\mathrm{Q}})$ and $\Psi^2 f(z_{\mathrm{Q}})$, we see that that they cannot be equal. We can conclude that there exists no such f.

We have thus found a way to use the representation ring together with the Adams operations Ψ^0 and Ψ^2 to distinguish between the groups D and Q. However, this was rather hard work and involved *ad hoc* arguments. We would like something that is easier and more systematic.

There is another fact that suggests that Adams operations are not the end of the story: It is known that the representation ring together with even *all* its Adams operations is not strong enough to distinguish all finite groups: Dade, in response to a question of Brauer, has shown that there are two groups of exponent 5 and order $5^7 = 78125$ that cannot be distinguished in this way.

An Idea from Topology

In order to produce a different refinement of the representation ring of a finite group, we let ourselves be inspired by topology. Every representation V of a group G gives rise to a vector bundle $\mathrm{E}G \times_G V$ over the classifying space $\mathrm{B}G$ of the group G. This association extends to a homomorphism

$$\mathrm{R}(G) \longrightarrow \mathrm{K}(\mathrm{B}G)$$

from the representation ring of the group G to the K-theory $\mathrm{K}(\mathrm{B}G)$ of the classifying space $\mathrm{B}G$. Atiyah has shown that this is almost an isomorphism: It is so after completion at the ideal of virtual representations of virtual dimension 0.

A while later, Bousfield introduced a refinement, 'united' K-theory, into algebraic topology. This theory unites (complex) K-theory, real K-theory, and self-conjugate K-theory. The latter, less well-known than the two former, was a new (in terms of those days) cohomology theory introduced in the (independent) theses of Anderson and Green. Anderson also studied the united K-theory of classifying spaces.

Bousfield's 'united' K-theory is the background for this project. Our aim has been to define 'united representation rings' as a representation theoretical invariant of finite groups.

United Representation Rings

A basis for the representation ring $\mathrm{R}(G)$, as an abelian group, is given by the set $\mathrm{Irr}(G)$ of irreducible characters of G. In fact, this basis is orthogonal with respect to the standard inner product

$$\langle \alpha \mid \beta \rangle = \frac{1}{|G|} \sum_g \overline{\alpha(g)} \beta(g).$$

The number $|\mathrm{Irr}(G)|$ of basis elements is the number of conjugacy classes of elements in G.

The representations of any group come in three different types: real, complex, and quaternionic. Let $\mathrm{Irr}_{\mathbb{C}}(G)$, $\mathrm{Irr}_{\mathbb{R}}(G)$ and $\mathrm{Irr}_{\mathbb{H}}(G)$ denote the subsets of $\mathrm{Irr}(G)$ consisting of complex, real and quaternionic representations, respectively. The characters in $\mathrm{Irr}_{\mathbb{R}}(G)$ and $\mathrm{Irr}_{\mathbb{H}}(G)$ are real valued, while the characters in $\mathrm{Irr}_{\mathbb{C}}(G)$ are genuinely complex valued. Therefore, the elements of $\mathrm{Irr}_{\mathbb{R}}(G)$ and $\mathrm{Irr}_{\mathbb{H}}(G)$ are invariant under conjugation, but the map $\chi \mapsto \overline{\chi}$ is a free involution on $\mathrm{Irr}_{\mathbb{C}}(G)$. In particular, this set has an even number of elements, and the set

$$\mathrm{Sum}_{\mathbb{C}}(G) = \{\, \chi + \overline{\chi} \mid \chi \in \mathrm{Irr}_{\mathbb{C}}(G) \,\}$$

has half the number of elements.

Let $\mathrm{RSC}(G)$ denote the subring of $\mathrm{R}(G)$ consisting of representations with real valued characters. In other words

$$\mathrm{RSC}(G) = \{\chi \in \mathrm{R}(G) \mid \chi(g) = \overline{\chi(g)} \text{ for all } g \in G\}.$$

A basis for the subring $\mathrm{RSC}(G)$, as an abelian group, is given by the disjoint union $\mathrm{Sum}_{\mathbb{C}}(G) \cup \mathrm{Irr}_{\mathbb{R}}(G) \cup \mathrm{Irr}_{\mathbb{H}}(G)$. Because Ψ^{-1} is conjugation, we see that the Adams operation Ψ^{-1} on a representation ring $\mathrm{R}(G)$

determines the subring $\mathrm{RSC}(G)$ of self-conjugate virtual representations.

Let $\mathrm{RO}(G)$ denote the subring of $\mathrm{RSC}(G)$ consisting of all real representations. If χ is a character of G, then the character $\chi + \overline{\chi}$ is obviously self-conjugate, but it turns out that it is even real. In particular, if χ has been self-conjugate in the first place, then the character 2χ is real. It follows that $\mathrm{RSC}(G)/\,\mathrm{RO}(G)$ is an elementary abelian 2-group. A basis for $\mathrm{RO}(G)$ should be given by the disjoint union $\mathrm{Sum}_{\mathbb{C}}(G) \cup \mathrm{Irr}_{\mathbb{R}}(G) \cup 2\mathrm{Irr}_{\mathbb{H}}(G)$.

We are now able to define what we mean by 'united representation rings'.

Definition. Let G be a finite group. The chain

$$\mathrm{RO}(G) \subseteq \mathrm{RSC}(G) \subseteq \mathrm{R}(G)$$

of subrings of the representation ring will be referred to as the *united representation ring* of the finite group G.

We will say that two groups G and H have isomorphic united representation rings if there exists an isomorphism $\mathrm{R}(G) \to \mathrm{R}(H)$ that maps $\mathrm{RSC}(G)$ isomorphically to $\mathrm{RSC}(H)$ and $\mathrm{RO}(G)$ isomorphically to $\mathrm{RO}(H)$. Formally, this is clearly stronger than having isomorphic representation rings, and we will now investigate how it turns out in practice.

The Example Revisited

If a group G has odd order, then it is easy to see, for instance by means of the Frobenius–Schur indicator discussed below, that the trivial representation is the only irreducible representation that is real. All other irreducible representations are complex, so that there are no irreducible representations that are quaternionic. For that reason we will consider only 2-groups from now on.

To see that the united representation ring is a truly stronger invariant of finite groups than the mere representation ring, we will look at the united representation rings of D and Q. Recall that these groups could not be distinguished using their ordinary representation rings. Since all the values in the character tables are real, none of the two groups have irreducible representations of complex type. We also see that all the 1-dimensional irreducible representations of both groups are real. However, the 2-dimensional irreducible representation of D is real, while the corresponding representation of Q is quaternionic. This implies that we have

$$\mathrm{RO}(\mathrm{D}) = \mathrm{RSC}(\mathrm{D}) = \mathrm{R}(\mathrm{D}),$$

while

$$\mathrm{RO}(\mathrm{Q}) \neq \mathrm{RSC}(\mathrm{Q}) = \mathrm{R}(\mathrm{Q}).$$

Hence, the two groups D and Q do not have isomorphic united representation rings, and one can use this structure to distinguish between the two groups. It is fair to say that this kind of argument is much easier than the one involving the second Adams operation above.

As we saw in the preceding paragraph, the united representation ring is a stronger invariant of a group than the ordinary representation ring. Thus, it is interesting to discuss *how much* stronger the united representation ring is. Can we find examples of non-isomorphic groups that cannot be distinguished by looking at their united representation rings? The answer to this question is yes, and the Hall–Senior groups $\Gamma_{26}a_2$ and $\Gamma_{26}a_3$ of order 64 serve as an example.

The Frobenius–Schur Indicator

One may wonder if, in analogy with the relation between ψ^{-1} on $\mathrm{R}(G)$ and the subring $\mathrm{RSC}(G)$, there can be a similar relation between the second Adams operation and the united representation ring. An indication of a connection between the two is given by a formula due to Frobenius–Schur: If χ is any class function on G, then the number

$$\mathrm{FS}(\chi) = \langle\, 1 \,|\, \Psi^2(\chi)\,\rangle$$

is called the *Frobenius–Schur indicator* of χ, and for an irreducible character χ we have

$$\mathrm{FS}(\chi) = \begin{cases} +1 & \text{if } \chi \text{ is real,} \\ 0 & \text{if } \chi \text{ is complex,} \\ -1 & \text{if } \chi \text{ is quaternionic.} \end{cases}$$

To turn this observation into a precise statement relating the the united representation ring and the second Adams operation, further investigation is needed.