Studforsk

Relative homology of the groups of non-singular matrices with binary coefficients

Erlend Due Børve

$n \backslash d$	0	1	2	3	4	5	6	7	8	$\cdots$
0	$\mathbb{Z}$	0	0	0	0	0	0	0	0	$\cdots$
1	0	0	0	0	0	0	0	0	0	$\cdots$
2	0	2	0	6	0	2	0	6	0	$\cdots$
3	0	0	4	2	0	14	2	2		
4	0	0	2	$2+2$	2	0	$2^{\oplus 3}$			
5	0	0	0	2						
$\vdots$	$\vdots$	$\vdots$	$\vdots$							

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Description

This project came about in an attempt to answer the following problem:

How do the groups $H_d(GL_n(\mathbb{F}_2))$ approach their limit $H_d(GL_\infty(\mathbb{F}_2))$?

The difference between the homology of $GL_{n+1}(\mathbb{F}_2)$ and $GL_n(\mathbb{F}_2)$ is measured by their relative homology groups, and we wanted to compute these at least for small n in small degrees. This report puts forward the how the topic has been investigated.

Preliminaries

Homology of groups

The n^{th} *homology* of a group G with coefficients in the G-module M is defined to be

$$H_n(G, M) = \operatorname{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, M)$$

More explicitly, if $P_\bullet(G)$ is a projective resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$, the group homology is equal to the homology of the complex $P_\bullet(G) \otimes_{\mathbb{Z}G} M$. For a given group G, let the $\mathbb{Z}$ be a G-module with trivial action, that is $gm = m$ for every $g \in G$ and for every $m \in \mathbb{Z}$. Let the map $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ send $\sum_{g\in G} a_g g$ to $\sum_{g\in G} a_g$. If we wish to compute homology with integral coefficients, a free resolution is usually constructed.

$$\ldots \longrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

Apply the functor $- \otimes_{\mathbb{Z}G} \mathbb{Z}$ to this resolution, and determine the homology of the result.

Having a closer look at the definition of group homology, it is seen that $H_0(G, \mathbb{Z}) = \mathbb{Z}$ for any group G. The first homology group is the abelianisation of G, namely $\frac{G}{[G,G]}$, and the second homology defines the *Schur multiplier* of a group.

If we are given a subgroup H inside G, the notion of relative homology may be defined. First, work out projective resolutions of H and G. Consider the sequence

$$0 \longrightarrow P_\bullet(H) \otimes_{\mathbb{Z}H} M \longrightarrow P_\bullet(G) \otimes_{\mathbb{Z}G} M \longrightarrow \frac{P_\bullet(G)\otimes_{\mathbb{Z}G} M}{P_\bullet(H)\otimes_{\mathbb{Z}H} M} \longrightarrow 0$$

which is exact because the map $P_\bullet(H) \otimes_{\mathbb{Z}H} M \to P_\bullet(G) \otimes_{\mathbb{Z}G} M$ is an injection. The homology of the quotient $P_\bullet(G, H; M) = \frac{P_\bullet(G)\otimes_{\mathbb{Z}G} M}{P_\bullet(H)\otimes_{\mathbb{Z}H} M}$ is what is meant by *relative homology*. In degree d, it is denoted by $H_d(G, H; M)$.

A consequence of the Snake Lemma is that a short exact sequence in chain complexes, like the one above, gives rise to a long exact sequence in homology.

$$\cdots \longrightarrow H_{d+1}(H; M) \longrightarrow H_{d+1}(G; M) \longrightarrow H_{d+1}(G, H; M) \longrightarrow$$
$$H_d(H; M) \longrightarrow H_d(G; M) \longrightarrow H_d(G, H; M) \longrightarrow$$
$$H_{d-1}(H; M) \longrightarrow H_{d-1}(G; M) \longrightarrow H_{d-1}(G, H; M) \longrightarrow \cdots$$

Spectral sequences

Suppose we have a *double complex* $C_{\bullet,\bullet}$, consisting of modules $\{C_{p,q}\}_{p,q\in\mathbb{Z}}$. There are vertical and horizontal morphisms, d_v and d_h, respectively, that make squares anti-commute. Should it also be assumed that $C_{p,q} = 0$ if at least one of p and q is negative, we have a *first quadrant double complex*. Visual interpretations are provided in Figures 1 and 2.

A *spectral sequence* is a set of *pages*, $E^r = \{E^r_{p,q}\}_{p,q\in\mathbb{Z}}$, that may be regarded as grids of modules. On the E^r-page there is a differential $\partial_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$.

Set $E^0_{p,q}$ to be $C_{p,q}$ and let ∂_0 be d_v.

$$\begin{array}{ccc}
\vdots & \vdots & \vdots \\
\downarrow{\scriptstyle\partial_0} & \downarrow{\scriptstyle\partial_0} & \downarrow{\scriptstyle\partial_0} \\
E^0_{p-1,q+1} & E^0_{p,q+1} & E^0_{p+1,q+1} \\
\downarrow{\scriptstyle\partial_0} & \downarrow{\scriptstyle\partial_0} & \downarrow{\scriptstyle\partial_0} \\
E^0_{p-1,q} & E^0_{p,q} & E^0_{p+1,q} \\
\downarrow{\scriptstyle\partial_0} & \downarrow{\scriptstyle\partial_0} & \downarrow{\scriptstyle\partial_0} \\
E^0_{p-1,q-1} & E^0_{p,q-1} & E^0_{p+1,q-1} \\
\downarrow{\scriptstyle\partial_0} & \downarrow{\scriptstyle\partial_0} & \downarrow{\scriptstyle\partial_0} \\
\vdots & \vdots & \vdots
\end{array}$$

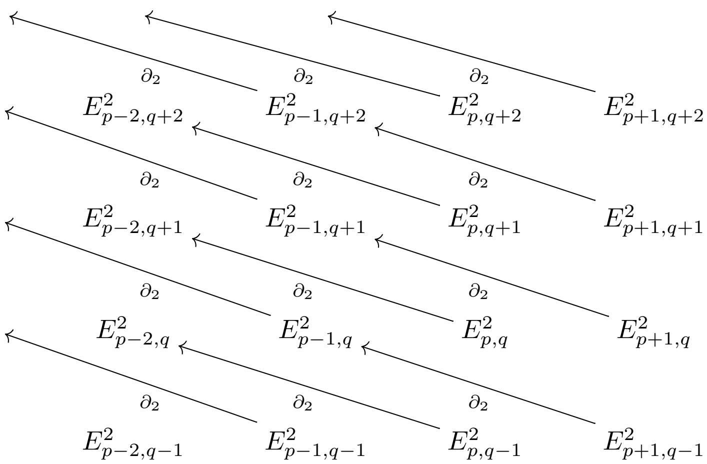
Next, let $E^1_{p,q}$ be the homology of degree q in position p. The differential $\partial_1 : E^1_{p,q} \to E_{p-1,q}$ is induced by d_h.

$$\cdots \xleftarrow[\partial_1]{} E^1_{p-1,q+1} \xleftarrow[\partial_1]{} E^1_{p,q+1} \xleftarrow[\partial_1]{} E^1_{p+1,q+1} \xleftarrow[\partial_1]{} \cdots$$

$$\cdots \xleftarrow[\partial_1]{} E^1_{p-1,q} \xleftarrow[\partial_1]{} E^1_{p,q} \xleftarrow[\partial_1]{} E^1_{p+1,q} \xleftarrow[\partial_1]{} \cdots$$

$$\cdots \xleftarrow[\partial_1]{} E^1_{p-1,q-1} \xleftarrow[\partial_1]{} E^1_{p,q-1} \xleftarrow[\partial_1]{} E^1_{p+1,q-1} \xleftarrow[\partial_1]{} \cdots$$

If we, yet again, take the homology of the resulting complexes, the E^2-page pops up.



This process might be carried out indefinitely. The limit of this process is called the E^∞-page.

Any first quadrant spectral sequence will reach the E^∞-page in a finite number of steps. The differentials will eventually grow so long that, for any given p and q, the sequence

$$E_{p+r,q-r-1} \xrightarrow{\partial_r} E_{p,q} \xrightarrow{\partial_r} E_{p-r,q+r-1}$$

must have $E_{p+r,q-r-1} = E_{p-r,q+r-1} = 0$, for sufficiently large values of r (the smallest of which depends on both p and q). Consequently, $E^r_{p,q} = E^{r+k}_{p,q}$, for every natural number k.

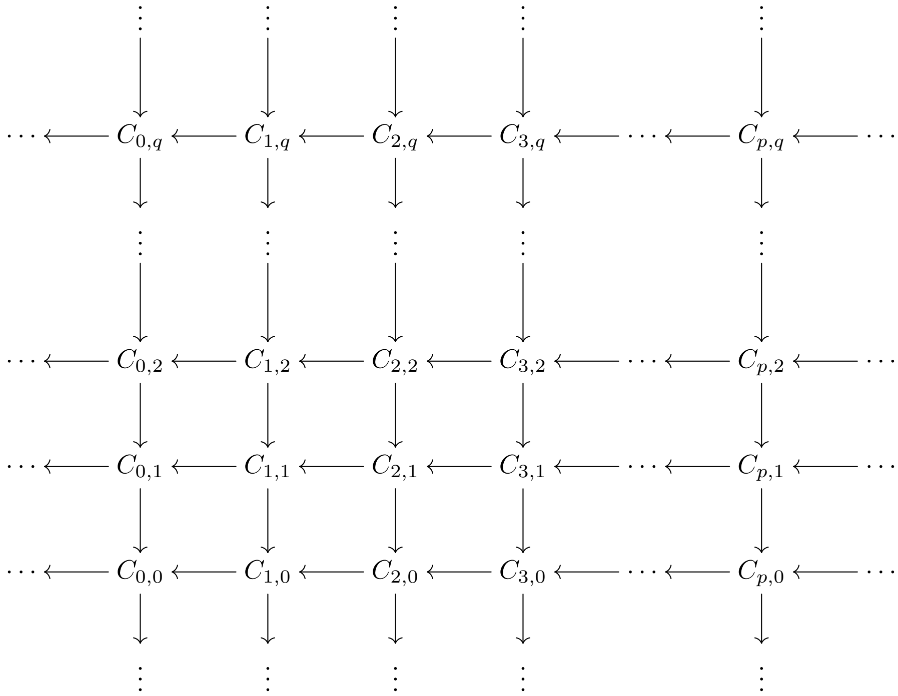


Figure 1: A double complex.

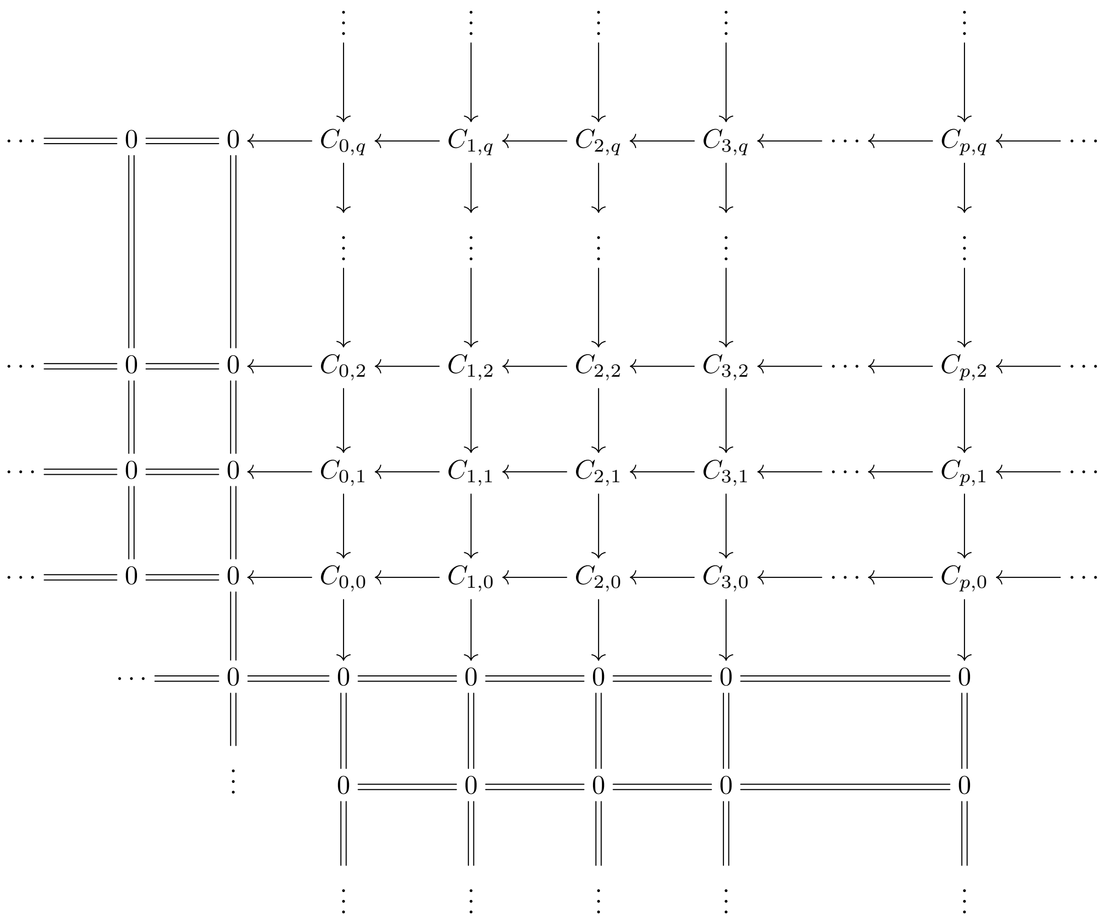


Figure 2: A first quadrant double complex.

The homology of the general linear groups over $\mathbb{F}_2$

In this project, the homology of the groups $GL_n(\mathbb{F}_2)$ has been of interest. The sizes of these groups grow dramatically with n (see Figure 3). The order of $GL_n(\mathbb{F}_2)$ is $\prod_{k=0}^{n-1}(2^n - 2^k)$, which is asymptotically equal to 2^{n^2}. This growth is clearly faster than exponential, and as matter of fact it is substantially faster than the growth of $n!$, the order of the symmetric groups Σ_n.

n	$\lvert GL_n(\mathbb{F}_2)\rvert$		
1	1		
2	6		
3	168		
4	20 160		
5	9 999 360	$\sim$	10^7
6	20 158 709 760	$\sim$	2.0×10^{10}
7	163 849 992 929 280	$\sim$	1.6×10^{14}
8	5 348 063 769 211 699 200	$\sim$	5.3×10^{18}
9	699 612 310 033 197 642 547 200	$\sim$	7.0×10^{23}
10	366 440 137 299 948 128 422 802 227 200	$\sim$	3.6×10^{29}

Figure 3: The orders of the groups $GL_n(\mathbb{F}_2)$ for $1 \leq n \leq 10$.

Due to the sheer magnitudes of the numbers in Figure 3, it is no surprise that the homology of $GL_n(\mathbb{F}_2)$ requires effort to attain, even for small values of n. The most difficult and time consuming part in the calculation of homology, is finding a projective resolution. Computers come handy in this regard. The programming language GAP is (in their own words) "[...] a system for computational discrete algebra, with particular emphasis on Computational Group Theory." The GAP-package HAP has useful commands that evaluate homology of a given group, in a given degree. However, when handling large groups like $GL_n(\mathbb{F}_2)$ with $n > 4$, considerable computing power and storage are required in order to gain useful results. Even when the order of $GL_n(\mathbb{F}_2)$ is not too excessive, like when $n = 4$, GAP spends an awful lot of time constructing a projective resolution, often exhausting the allocated storage in the process.

The following isomorphisms are worthy of mention.

$$\begin{aligned} GL_0(\mathbb{F}_2) \cong GL_1(\mathbb{F}_2) &\cong C_1 \\ GL_2(\mathbb{F}_2) &\cong \Sigma_3 \\ GL_3(\mathbb{F}_2) &\cong PSL_2(\mathbb{F}_7) \\ GL_4(\mathbb{F}_2) &\cong A_8 \end{aligned}$$

In general $GL_n(\mathbb{F}_2) \cong {}^2A_n(2^2)$, the unitary Steinberg group of rank n over $\mathbb{F}_2$. These groups are simple when n is greater than 2, and the Schur multipliers are known to be 0 for all n greater than 4.

The groups $GL_0(\mathbb{F}_2)$ and $GL_1(\mathbb{F}_2)$ are trivial. The homology of the symmetric group Σ_3 is not terribly hard to find, using the definition.

$$H_d(\Sigma_3; \mathbb{Z}) = \begin{cases} \mathbb{Z} & d = 0 \\ \mathbb{Z}/2 & d \equiv 1 \bmod 4 \\ 0 & d \neq 0 \text{ and even} \\ \mathbb{Z}/6 & d \equiv 3 \bmod 4 \end{cases}$$

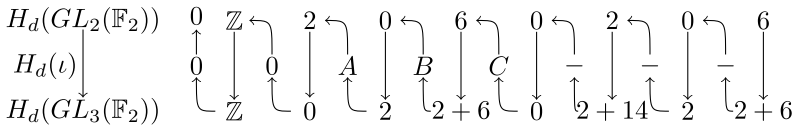
For large values of n, no well-behaved periodicity of this kind occurs. The remainder of the calculations of (absolute) homology have been performed by GAP. Figure 4 displays the homology of $GL_n(\mathbb{F}_2)$, as far as time and storage has permitted. In order to save space, the expression $2 + 6$ denotes the group $\mathbb{Z}/2 \times \mathbb{Z}/6$, for instance.

Nonetheless, it was not the absolute, but relative homology with which this project was concerned. Some of the relative homology groups are easily deduced using the long exact sequence in homology. When n is small, specifically when n is less than 3, the group $H_d(GL_n(\mathbb{F}_2), GL_{n-1}(\mathbb{F}_2))$ is found in such a manner, for any choice of d. However, when n is greater than or equal to 3, more machinery is needed to genuinely examine the situation.

In certain cases, relative homology could be found by computer. The cases of $H_d(GL_3(\mathbb{F}_2), GL_2(\mathbb{F}_2))$, when $d \in \{2, 3, 4\}$ (indicated by A, B and C, respectively, in the diagramme below) provide examples of this.

d	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
GL_0	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	$\ldots$
GL_1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	$\ldots$
GL_2	$\mathbb{Z}$	2	0	6	0	2	0	6	0	2	0	$\ldots$
GL_3	$\mathbb{Z}$	0	2	$2+6$	0	$2+14$	2	$2+6$	$2+2$	$2+2$	2	
GL_4	$\mathbb{Z}$	0	2	$2+6$	2	$2+14$	$2^{\oplus 4}$					
GL_5	$\mathbb{Z}$	0	0									
$\vdots$	$\vdots$	$\vdots$	$\vdots$									

Figure 4: The homology of the general linear groups for small n in small degrees.



The general linear groups (over any field) carry a fortunate multiplicative structure, of which it may be taken advantage to determine $H_d(GL_3(\mathbb{F}_2), GL_2(\mathbb{F}_2))$. There is a canonical copy of the group $GL_k(F) \times GL_l(F)$ inside $GL_{k+l}(F)$ in the following way:

$$GL_k(F) \times GL_l(F) \cong \begin{pmatrix} GL_k(F) & 0 \\ 0 & GL_l(F) \end{pmatrix} \subset GL_{k+l}(F)$$

A GAP script has been produced, that calculates the relative homology of $GL_{k+l}(\mathbb{F}_2)$ and $GL_k(\mathbb{F}_2)$, using this canonical injection, with $l = 1$. It finds:

$$H_2(GL_3(\mathbb{F}_2), GL_2(\mathbb{F}_2)) \cong \mathbb{Z}/4$$
$$H_3(GL_3(\mathbb{F}_2), GL_2(\mathbb{F}_2)) \cong \mathbb{Z}/2$$
$$H_4(GL_3(\mathbb{F}_2), GL_2(\mathbb{F}_2)) \cong 0$$

The same script has identified (using far more computing power) the groups $H_d(GL_4(\mathbb{F}_2), GL_3(\mathbb{F}_2))$ for $d \in \{1, 2, 3\}$.

$$H_2(GL_4(\mathbb{F}_2), GL_3(\mathbb{F}_2)) \cong 0$$
$$H_3(GL_4(\mathbb{F}_2), GL_3(\mathbb{F}_2)) \cong \mathbb{Z}/2$$
$$H_4(GL_4(\mathbb{F}_2), GL_3(\mathbb{F}_2)) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$$

Merely for the sake of convenience, let $GL_{-1}(\mathbb{F}_2)$ be the "empty group". It has trivial homology in any degree, even in degree 0. Figure 5 puts forward a summary of these results.

d	0	1	2	3	4	5	6	7	8	$\ldots$
$(GL_0(\mathbb{F}_2), GL_{-1}(\mathbb{F}_2))$	$\mathbb{Z}$	0	0	0	0	0	0	0	0	$\ldots$
$(GL_1(\mathbb{F}_2), GL_0(\mathbb{F}_2))$	0	0	0	0	0	0	0	0	0	$\ldots$
$(GL_2(\mathbb{F}_2), GL_1(\mathbb{F}_2))$	0	2	0	6	0	2	0	6	0	$\ldots$
$(GL_3(\mathbb{F}_2), GL_2(\mathbb{F}_2))$	0	0	4	2	0					
$(GL_4(\mathbb{F}_2), GL_3(\mathbb{F}_2))$	0	0	2	$2+2$						
$(GL_5(\mathbb{F}_2), GL_4(\mathbb{F}_2))$	0	0	0							
	$\vdots$	$\vdots$	$\vdots$							

Figure 5: The relative homology of the general linear groups for small n in small degrees.

Killing of 2-torsion

To get a grip of what the remaining relative homology groups may be like, the spectral sequence structure among these groups should be exploited. This structure is laid out in Figure 6. Red entries indicate educated guesses, with the intention of killing all 2-torsion in the limit $GL_\infty(\mathbb{F}_2)$. Note that q corresponds to n, and that $p = n + d$, where d is the degree. The spectral sequence in (d, n)-coordinates is found in Figure 7.

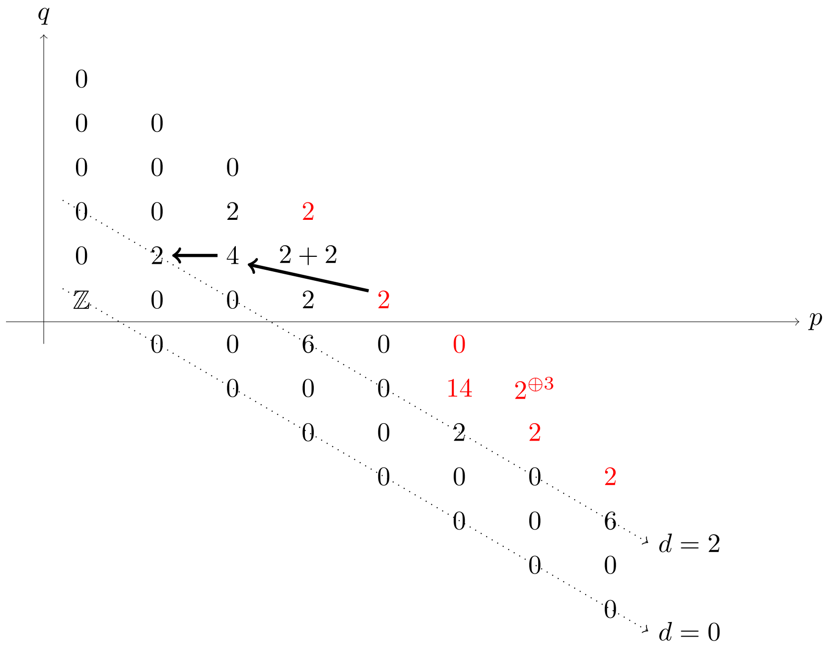


Figure 6: The spectral sequence structure among the relative homology groups, with examples of ∂_1- and ∂_2-differentials.

Quillen has shown that $H_d(GL_\infty(\mathbb{F}_p);\mathbb{Z})$ contains no p-torsion [4]. In particular, the homology of the limit $GL_\infty(\mathbb{F}_2)$ contains no 2-torsion, and thus all 2-torsion must be killed by differentials. The educated guesses in Figure 6 arise thusly. (It is perfectly possible that the guessed groups are somewhat larger than presumed, should some unknown 2-torsion not be accounted for.)

Homological stability

The homology of the symmetric groups is much better understood, and indeed much more well-behaved than of the general linear groups. In the early 60s, Nakaoka produced a theorem on the homological stability of these groups [5].

Theorem 1. *$H_m(\Sigma_{n-1}) = H_m(\Sigma_n)$ for $m < \frac{n}{2}$.*

In other words, the relative homology group $H_m(\Sigma_{n-1}, \Sigma_n; \mathbb{Z})$ is 0 for $m < \frac{n}{2}$. An elementary proof of this theorem has since been found by Kerz, using the complex of injective words [6]. Another convenient fact is that the maps $H_d(\iota) : H_d(S_n) \to H_d(S_{n+1})$ are split injections, meaning that the sequence of groups $\{H_d(S_i;\mathbb{Z})\}_{i=\mathbb{N}}$ "increases monotonously" ($H_d(S_{i+1};\mathbb{Z}) \cong H_d(S_i;\mathbb{Z}) \oplus G$, for some group G) and that is "bounded" due to stability. The relative homology groups are in this case easily found by taking quotients.

Such simplicity does not apply to the general linear groups. The induced maps

$$H_d(\iota) : H_d(GL_n(\mathbb{F}_2)) \to H_d(GL_{n+1}(\mathbb{F}_2))$$

are not always split injections, (in fact, $H_d(\iota) : H_d(GL_2(\mathbb{F}_2)) \to H_d(GL_3(\mathbb{F}_2))$ is not even an injection). Fortunately, there is still a stability result to employ. Nearly twenty years after Nakaoka's result was published, a similar theorem was proved for the general linear groups, attributed to Maazen [7] and Charney [8].

Theorem 2. *$H_m(GL_{n-1}(F)) = H_m(GL_n(F))$ for $m < \frac{n}{2}$ for all fields F.*

Hence, the table of relative homology may be separated into two regions. Whenever $m < \frac{n}{2}$, the above theorem ensures that $H_m(GL_{2n}(\mathbb{F}_2), GL_{2n-1}(\mathbb{F}_2)) = 0$, and whenever $m \geq \frac{n}{2}$, there could be non-zero entries.

The spectral sequence presented in Figure 6 is not a first quadrant spectral sequence, but homological stability shows that it also converges in this sense. For a given $H_d(GL_n(\mathbb{F}_2), GL_{n-1}(\mathbb{F}_2))$ it may be hit by differentials with domain $H_{d+1}(GL_{n+k+1}(\mathbb{F}_2), GL_{n+k}(\mathbb{F}_2))$, where k is a natural number (including 0). For large enough k, the domain of the differentials land within the zone of stability, and hence a similar situation to the first quadrant case is observed.

Conclusion and conjectures

A final table of relative homology groups is put forward in Figure 7. It is identical to Figure 6, up to a transformation.

$n \backslash d$	0	1	2	3	4	5	6	7	8	$\cdots$
0	$\mathbb{Z}$	0	0	0	0	0	0	0	0	$\cdots$
1	0	0	0	0	0	0	0	0	0	$\cdots$
2	0	2	0	6	0	2	0	6	0	$\cdots$
3	0	0	4	2	0	14	2	2		
4	0	0	2	$2+2$	2	0	$2^{\oplus 3}$			
5	0	0	0	2						
$\vdots$	$\vdots$	$\vdots$	$\vdots$							

Figure 7: Table of the relative homology groups of $GL_n(\mathbb{F}_2)$.

Although it known for certain what occurs below the line $d = \frac{n}{2}$, it is not quite as certain what groups live along it. An interesting conjecture is that $H_d(GL_{2n}(\mathbb{F}_2), GL_{2n-1}(\mathbb{F}_2))$ are all non-zero. This would mean that the Maazen–Charney stability gives a strict bound. To prove this, it is sufficient to prove that for every positive integer p, $E^2_{p,p}$ is non-zero in the spectral sequence. Additionally, it may be asked whether there are differentials of arbitrary length.

Acknowledgments

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