Project: Support identification for linear equations with sparsity constraints

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Background: In this project we aim at developing and analyzing techniques for multi-penalty regularization in spaces of sparsely represented functions. This research is inspired by several recent developments in regularization theory, optimization, and signal processing, and has applications in medical image analysis, audio analysis, and data mining. In real-life applications, it is often not possible to observe the quantity of interest directly; instead, one only has access to possibly high dimensional and noisy indirect measurement data. Additionally, the original signal to be recovered may be affected by its own noise. In this case, the reconstruction problem can be understood as an inverse problem, where the solution $x^\dagger$ consists of two (or more) components of different nature, the relevant signal and its noise, that have to be separated.

Mathematical framework: We consider the problem of solving a linear system $A(u+v) = b$, where u, v are the two components of the solution $x^\dagger$ which we wish to identify and to separate, $b \in \mathbb{R}^m$ is the given data vector, and $A \in \mathbb{R}^{m\times n}$ is a full rank matrix. In the case where $n > m$, that is, the parameter space $\mathbb{R}^n$ has a higher dimension than the data space $\mathbb{R}^m$, this equation is not uniquely solvable. However, a unique solution is possible if the matrix A has a sufficiently 'nice' structure and one knows *a priori* that one of the components of the 'true' solution $x^\dagger$, say u, is sparse (the vector u has a small number of non-zero entries), and another component v, interpreted as noise, has bounded coefficients, i.e., a small maximum norm. In this case, one can try to reconstruct u (and v) by solving, for some regularization parameter $\alpha > 0$, the constrained optimization problem

$$(1) \qquad \|u\|_1 + \alpha\|v\|_\infty \to \min \qquad \text{subject to} \qquad A(u+v) = b.$$

In general, this will not lead to an exact reconstruction of u, but in practice one often only needs to reconstruct the support of u, i.e., the indices i for which $u_i \neq 0$.

Problem:

(1) Try to find reasonable conditions for the matrix A, the regularization parameter α, and the noise level that guarantee that the support of u is reconstructed exactly whenever it is sufficiently small.

(2) If also the data b is perturbed, one can replace the strict constraint $A(u+v) = b$ by a relaxed constraint $\|A(u+v) - b\|_2 \leq \delta$ for some parameter $\delta > 0$ or, equivalently, solve an optimization problem of the form

$$\|u\|_1 + \alpha\|v\|_\infty + \frac{\lambda}{2}\|A(u+v) - b\|_2^2 \to \min$$

for some suitable parameter $\lambda \geq 0$. Can the results also be generalized to such a situation?

(3) *(optional)* Implement the solution of (1) and test the method on suitable matrices A.

Prerequisites: The student should have successfully completed the course "linear methods"; the course "optimization theory" is recommended in particular for the (optional) part (3).

Training: At the start of the project, the student will receive an introduction to inverse problems and compressed sensing.

Time Frame: The project is estimated to take ca. 100 hours of work.