TMA4230 Functional Analysis, 31 May 2005

Suggested solution

Text in small type, like this, is not part of the solution, but rather comments on the solution.

Problem 1

a) The *closed graph theorem* states that, if X and Y are Banach spaces and $T\colon X \to Y$ is a linear operator with closed graph in $X \times Y$, then T is bounded.

An operator with a closed graph is itself called closed, so an even briefer statement of the theorem is possible and acceptable. Kreyszig considers the case where T is defined only on a closed subspace of X. This is only *apparently* more general, since a closed subspace of a Banach space is itself a Banach space. But of course, this formulation is also acceptable. It is necessary for both X and Y to be Banach spaces, as counterexamples can be constructed if either of them is not complete.

b) First, $\ker P$ is a closed subspace: This is true of the kernel of *every* bounded linear operator, whether or not they are projections. From $P^2 = P$ we get $PPx = Px$, so that $Py = y$ if $y = Px \in \operatorname{im} P$. It follows that $\operatorname{im} P = \ker(I - P)$, which is a closed subspace. If $y \in \operatorname{im} P \cap \ker P$ then $y = Py = 0$, so $y = 0$. Thus $\operatorname{im} P \cap \ker P = \{0\}$. And we can always write $x = Px + (I - P)x$ with $Px \in \operatorname{im} P$ and $(I - P)x \in \ker P$, so that $X = \operatorname{im} P + \ker(I - P)$.

c) By assumption, every $x \in X$ is uniquely written $x = y + z$ with $y \in Y$ and $z \in Z$. If P is a projection with image Y and kernel Z, then we are forced to have $Py = y$ and $Pz = 0$, so $Px = P(y + z) = y$. Moreover, this is well defined P because of the uniqueness of the decomposition of x.

To prove the linearity of P is straightforward, and I'll skip it here.

To prove that P is bounded, we use the closed graph theorem. If $x \in X$ we write as before $x = y + z$ with $y \in Y$, $z \in Z$, and note that $Px = y$. Thus we get a pair $(x, y) = (y + z, y)$ in the graph of P, and we note that every member of the graph of P can be written this way. Put differently, a pair (x, y) belongs to the graph of P if and only if $y \in Y$ and $x - y \in Z$.

Let now (x_n, y_n) belong to the graph of P for each n, and assume $x_n \to x$ and $y_n \to y$ in the norm topology, i.e., $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$.

Now, since Y is closed, $y \in Y$ is an immediate consequence. And since $x_n - y_n \to x - y$ and $x_n - y_n \in Z$, and moreover Z is closed, we get $x - y \in Z$. Thus the limit (x, y) belongs to the graph of P. We have proved that P has a closed graph, and therefore the boundedness of P.

Problem 2

a) If A is an algebra with unit, $x \in A$, and p is a polynomial, then $\sigma\big(p(x)\big) = p\big(\sigma(x)\big)$.

b) We apply the identity of the previous question. Since now $p(x) = 0$, we get $\sigma\big(p(x)\big) = \{0\}$, so that $p\big(\sigma(x)\big) = \{0\}$. That is, $\sigma(x)$ is contained in the zero set of p, which is a finite set.

Special case $p(t) = t^4 - 1$: The zero set of p is $\{-1, +1, -i, +i\}$.

Problem 3

a) Any bounded self-adjoint operator T on a Hilbert space H can be written

$$T = \int_{\mathbb{R}} \lambda \, dE_\lambda$$

where $(E_\lambda)_{\lambda\in\mathbb{R}}$ is the *spectral family* of T.

The spectral family satisfies these properties: Each E_λ is a self-adjoint projection, $E_\lambda \le E_\mu$ if $\lambda < \mu$, $E_\lambda = 0$ for $\lambda < m$ and $E_\lambda = I$ for $\lambda \ge M$ where m and M are the smallest and largest elements of $\sigma(T)$, and E_λ is the strong operator limit of E_μ as $\mu \to \lambda^+$.

The integral is the limit – in the norm topology – of the Riemann–Stieltjes sums

$$\sum_{k=1}^{n} \lambda_k^* (E_{\lambda_k} - E_{\lambda_{k-1}})$$

where $\lambda_0 < \lambda_1 < \cdots < \lambda_n$, $\lambda_0 < m$, $\lambda_n \ge M$, $\lambda_{k-1} \le \lambda_k^* \le \lambda_k$ for $k = 1, \ldots, n$, and the limit is taken as the norm $\max_k |\lambda_k - \lambda_{k-1}|$ goes to zero.

Finally,

$$f(T) = \int_{\mathbb{R}} f(\lambda) \, dE_\lambda$$

for any continuous function f on $\sigma(T)$.

The latter definition requires values of f outside $\sigma(T)$, for example in any gaps in the spectrum. This is remedied by expanding the definition of f in a suitable way, for example by making it affine in each such gap. The exact manner of extension is unimportant, in the limit.

b) It is known that E_λ is constant in each subinterval of the complement of the spectrum. There are $n+1$ such intervals, yielding $n+1$ different values of E_λ. By the continuity from the right of the spectral map, the value of each E_{λ_k} is the value of E_λ in the interval to the right of λ_k. Thus

$$E_\lambda = \begin{cases} 0 & \lambda < \lambda_1, \\ E_{\lambda_k} & \lambda_k \le \lambda < \lambda_{k+1}, \quad k = 1, \ldots, n-1, \\ I & \lambda \ge \lambda_n. \end{cases}$$

If we write $P_1 = E_{\lambda_1}$ and $P_k = (E_{\lambda_k} - E_{\lambda_{k-1}})$ for $k = 2, \ldots, n$ the integral becomes

$$T = \int_{\mathbb{R}} \lambda \, dE_\lambda = \sum_{k=1}^{n} \lambda_k P_k.$$

The following question was not supposed to be here. It had been replaced by a different question, but then an early version was delivered to the exam office by mistake.

The students were told around 11:30, and told that they could skip this question and still get a full score.

c) The definition of U_t becomes

$$U_t = e^{itT} = \int_{\mathbb{R}} e^{it\lambda} \, dE_\lambda.$$

The adjoint becomes

$$U_t^* = e^{-itT} = \int_{\mathbb{R}} e^{-it\lambda} \, dE_\lambda.$$

We compute the product by multiplying the integrands pointwise: $e^{it\lambda}e^{-it\lambda} = 1$ and integrating: $U_t U_t^* = U_t^* U_t = \int_{\mathbb{R}} dE_\lambda = I$. Similarly $U_s U_t = \int_{\mathbb{R}} e^{is\lambda} e^{it\lambda} \, dE_\lambda = \int_{\mathbb{R}} e^{i(s+t)\lambda} \, dE_\lambda = U_{s+t}$. Finally, to compute the derivative consider

$$\frac{U_s - U_t}{s - t} = \int_{\mathbb{R}} \frac{e^{is\lambda} - e^{it\lambda}}{s - t} \, dE_\lambda$$

and let $s \to t$. Then the integrand converges to the derivative $i\lambda$. Moreover the convergence is uniform, so we can take the limit inside the integral and get $dU_t/dt = iTU_t$. (The factor i was missing from the problem statement.)