Exercise set B

Some exercises for TMA4230 Functional analysis

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Exercise B.1. Let $0 < p < 1$ be given. Consider $L^p = L^p[0,1]$: The space of L^p functions on the unit interval with Lebesgue measure. The L^p "norm" is not really a norm in this case, since it fails to satisfy the triangle inequality. Instead, we can make a metric d_p on L^p by

$$d_p(u,v) = \|u-v\|_p^p = \int_0^1 |u-v|^p\,dx.$$

Show that d_p is a metric. *Hint*: It is enough to show $|u+v|^p \le |u|^p + |v^p|$ and then integrate this inequality. Since $|u+v| \le |u|+|v|$, it is sufficient to show the inequality when $u, v \ge 0$. That is, show that $(u+v)^p \le u^p + v^p$ for $u, v \ge 0$. This is an equality for $v = 0$. Differentiate wrt v.

It can be shown that L^p is complete in this metric, and the topology induced by this metric makes L^p into a topological vector space as well. (You are not expected to show this, but you are welcome to do it anyway.)

The main purpose of this problem is to show that the only continous linear functional on L^p is the zero functional. (Why does this prove that L^p is not locally convex?)

First, show that a linear functional f on L^p is continuous if and only if it is bounded, in the sense that $\sup_{\|u\|_p=1}|f(u)| \le \infty$. Next, note that because of this, if there exists a nonzero continuous linear functional on L^p, there is one such functional for which

$$\sup_{\|u\|_p=1} |f(u)| = 1.$$

Pick any $u \in L^p$ with $\|u\|_p = 1$. Now, if you can write $u = u_1 + u_2$ where $u_1 u_2 = 0$, then $\|u\|_p^p = \|u_1\|_p^p + \|u_2\|_p^p$. (Why?) Explain how to take advantage of the non-discrete nature of the Lebesgue measure to split up u in this way so that $\|u_1\|_p^p = \|u_2\|_p^p = \frac{1}{2}\|u\|_p^p = \frac{1}{2}$. Conclude that $|f(u)| \le 2^{1-1/p}$, and obtain a contradiction.

Exercise B.2. A *ordered vector space* is a real vector space with a partial ordering $\le$ so that: Whenever $x \le y$ then $cx \le cy$ for any real number $c \ge 0$; and $x + v \le y + v$ for any $v \in X$.

If X is an ordered vector space and $X^+ = \{x \in X : x \ge 0\}$, show that $0 \in X^+$, $cx \in X^+$ whenever $c \ge 0$ is real and $x \in X^+$, $X^+ \cap (-X^+) = \{0\}$, $x + y \in X^+$ whenever $x, y \in X^+$, and X^+ is convex. (Also show that the two final conditions are equivalent, given the first three.) We say that X^+ is a *proper convex cone* in X.

Show also that if X^+ is a proper convex cone in X, we can make X into an ordered vector space by saying $x \le y \Leftrightarrow y - x \in X^+$.

Some examples of ordered vector spaces include: All sequence spaces and all function spaces (with real-valued sequences and functions), and the space of bounded Hermitian operators on a Hilbert space, where $S \le T$ if and only if $T - S$ is nonnegative definite. For examples of order unit spaces, as will be defined next, let e be constant function on any function space consisting of bounded functions, or the identity operator in the space of bounded Hermitian operators.

Exercise B.3. An *order unit space* is an ordered vector space with an *order unit* $e \in X^+$ which is supposed to satisfy two requirements: First, if $x \in X$ there exists some real number c with $-ce \le x \le ce$, and second, if $x \in X$ and $cx \le e$ for all $c \ge 0$ then $x \le 0$.

From now on through the remaining problems, let X be an order unit space with order unit e.

Show that if we define

$$\|x\| = \inf\{c\colon -ce \le x \le ce\} \qquad (x \in X)$$

then $\|\cdot\|$ is in fact a norm, and $-\|x\|e \le x \le \|x\|e$.

Exercise B.4. Define the *state space* S of X to be the set of linear functionals

$$S = \{f \in X^*\colon f(x) \ge 0 \text{ for all } x \ge 0, \text{ and } f(e) = 1\}.$$

Show that $\|f\| = 1$ for all $f \in S$, and S is compact in the weak* topology.

Exercise B.5. Show that if $x \in X$ then $x \ge 0$ if and only if $f(x) \ge 0$ for all $f \in S$.
Hint: If $x \not\ge 0$, use Hahn–Banach to separate x from X^+.

Exercise B.6. Show that $\|x\| = \sup\{|f(x)|\colon f \in S\}$.
Hint: Assume that $(\|x\| - \epsilon)e - x \not\ge 0$ for every $\epsilon > 0$. (If not, replace x by $-x$.) Apply the previous exercise.

Exercise B.7. Show that the unit ball of X^* is $X_1^* = \operatorname{co}(S \cup -S)$.
Hint: If not, separate some point in X_1^* from $\operatorname{co}(S \cup -S)$ using the Hahn–Banach theorem, and obtain a contradiction.