Solution set 1

to some problems given for TMA4230 Functional analysis

2005–02–03

Problem 4.2.3. The task is to show that $p(x) = \overline{\lim}_{n\to\infty} x_n$ is a sublinear functional on the real version of ℓ^∞. Remember the definition $\overline{\lim}_{n\to\infty} x_n = \lim_{n\to\infty} \sup_{k\geq n} x_k$.

First, assuming $\alpha \geq 0$ we find

$$p(\alpha x) = \overline{\lim_{n\to\infty}}\, \alpha x_n = \lim_{n\to\infty} \sup_{k\geq n} \alpha x_k = \lim_{n\to\infty} \alpha \sup_{k\geq n} x_k = \alpha \lim_{n\to\infty} \sup_{k\geq n} x_k = \alpha \overline{\lim_{n\to\infty}}\, \alpha x_n = \alpha p(x)$$

Second, we find

$$\begin{aligned} p(x+y) &= \overline{\lim_{n\to\infty}}\, \alpha(x_n + y_n) = \lim_{n\to\infty} \sup_{k\geq n} (x_k + y_k) \leq \lim_{n\to\infty} \Bigl(\sup_{k\geq n} x_k + \sup_{k\geq n} y_k\Bigr) \\ &= \lim_{n\to\infty} \sup_{k\geq n} x_k + \lim_{n\to\infty} \sup_{k\geq n} y_k = \overline{\lim_{n\to\infty}}\, x_k + \overline{\lim_{n\to\infty}}\, y_k = p(x) + p(y). \end{aligned}$$

Problem 4.2.4. If p is sublinear then $p(0) = p(0 \cdot 0) = 0p(0) = 0$.[1] Next, $p(0) = p\bigl(x + (-x)\bigr) \leq p(x) + p(-x)$. Substituting $p(0) = 0$ and subtractin $p(x)$ we get $-p(x) \leq p(-x)$.

Problem 4.2.5. Let $x, y \in M$. I.e., $p(x) \leq \gamma$ and $p(y) \leq \gamma$. Assume $0 \leq \alpha \leq 1$. Then

$$p\bigl(\alpha x + (1-\alpha)y\bigr) \leq p\bigl(\alpha x\bigr) + p\bigl((1-\alpha)y\bigr) = \alpha p(x) + (1-\alpha)p(y) \leq \alpha\gamma + (1-\alpha)\gamma = \gamma,$$

so that $\alpha x + (1-\alpha)y \in M$. The requirement $\gamma > 0$ in the problem is in fact not needed.

Problem 4.2.10. To show $-p(-x) \leq \tilde{f}(x) \leq p(x)$ where $\tilde{f}$ is linear and p is sublinear, it is enough to show $\tilde{f}(x) \leq p(x)$ for all x. For then we must also have $f(-x) \leq p(-x)$, which after multiplication with -1 becomes $f(x) \geq -p(-x)$.

To obtain the required $\tilde{f}$, then, it is enough to start with the zero functional $f(0) = 0$ on the trivial subspace $\{0\}$. It obviously satisfies $f(x) \leq p(x)$ for $x \in \{0\}$, so by the Hahn–Banach theorem it has an extension $\tilde{f}$ satisfying $\tilde{f}(x) \leq p(x)$ for all x.

Remark. When applied to the example of problem 4.2.3, this shows the existence of a linear functional $\tilde{f}$ on ℓ^∞ so that

$$\underline{\lim_{x\to\infty}}\, x_n \leq \tilde{f}(x) \leq \overline{\lim_{x\to\infty}}\, x_n \qquad (x \in \ell^\infty).$$

Such a linear functional is impossible to describe explicitly. However, if you are given a free ultrafilter $\mathcal{U}$ on the natural numbers (another object with no explicit description), then one can construct a linear functional $\tilde{f}$ satisfying the above inequalities by

$$\tilde{f}(x) = \lim_{n\to\mathcal{U}} x_n \qquad (x \in \ell^\infty).$$

If this remark makes no sense to you right now, don't worry about it. It will make sense later.

Problem 4.3.3. (We drop the tilde on $\tilde{f}$.) We are given a *real* linear functional f on a *complex* vector space X, satisfying $f(ix) = if(x)$ for every $x \in X$. The task is to show that f is in fact *complex* linear.

In other words, we must show that $f(\gamma x) = \gamma f(x)$ for complex γ. Write $\gamma = \alpha + i\beta$ with α and β real. Using the real linearity of f together with the given identity $f(ix) = if(x)$ we find

$$f(\gamma x) = f(\alpha x + i\beta x) = f(\alpha x) + f(i\beta x) = f(\alpha x) + if(\beta x) = \alpha f(x) + i\beta f(x) = \gamma f(x).$$

Problem 4.3.11. Assume, on the contrary, that $x \neq y$. Create a linear functional f on the one-dimensional space spanned by $x - y$ so that $f(x-y) = 1$. Then f is bounded.[2] Thus by Hahn–Banach there is a bounded extension $\tilde{f}$ on X. Since $\tilde{f}(x-y) = f(x-y) = 1$, $\tilde{f}(x) \neq \tilde{f}(y)$, which contradicts the assumption that no bounded linear functional can tell the difference between x and y.

[1] Watch out: Some zeros in that calculation signify the *scalar* 0, whereas others stand for the zero *vector*.

[2] *Any* linear functional on a finite-dimensional space is bounded. But of course, we can easily compute $\|f\| = 1/\|x-y\|$.