Exercise set 3

For TMA4230 Functional analysis

2006–04–01

Exercise 3.1. Let X and Y be topological spaces. Show that a function $f\colon X \to Y$ is continuous if and only if $f^{-1}(F)$ is closed in X for every closed $F \subseteq Y$.

Exercise 3.2. We have defined compactness of X in terms of open covers of X, which are sets of open subsets of X covering X.

Instead, consider now a subset K of a topological space X. Then K with the topology inherited from X is a topological space in its own right, so we can ask if K it compact or not.

If we define an open cover of K to be a set of open subsets of X whose union contains K, prove that K is compact if and only if every open cover of K has a finite subcover (of K). (The point here is that compactness of K is defined in terms of open covers of K, as consisting of subsets of K which are open in the inherited topology.)

Exercise 3.3. Show that any closed subset of a compact space is compact.

Exercise 3.4. Show that any compact subset of a Hausdorff space is closed. (But note: If X is the space $\{0,1\}$ with topology $\{\emptyset,\{0\},\{0,1\}\}$ then $\{0\}$ is compact but not closed. Of course, X is not Hausdorff either.)

Exercise 3.5. Let X be a set and $\mathscr{B}$ a set of subsets of X. Let $\mathscr{B}'$ be the set of all finite intersections from $\mathscr{B}$:

$$\mathscr{B}' = \{B_1 \cap \cdots \cap B_n \colon B_1,\ldots,B_n \in \mathscr{B}; n = 0,1,2,\ldots\}$$

with the understanding that $B_1 \cap \cdots \cap B_n = X$ when $n = 0$. Let $\mathscr{T}$ consist of all possible unions of members of $\mathscr{B}'$. Show that $\mathscr{T}$ is a topology; in fact, it is the weakest topology containing $\mathscr{B}$. It is said to be the topology *generated* by $\mathscr{B}$. Also, $\mathscr{B}$ is said to be a *basis* for T.

Exercise 3.6. Let X be a real topological vector space, not necessarily locally convex. Show that the intersection of all convex neighbourhoods of 0 is a (vector) subspace of X. Call this space Z.

Show that any continous linear functional on X vanishes on Z (in other words, if f is a continuous linear functional and $z \in Z$ then $f(z) = 0$.) *Hint*: If V is any neighbourhood of 0, then so is αV whenever $\alpha > 0$.

Show that Z is closed. In fact, show that if $x \in X \setminus Z$ then there is a convex neighbourhood of x that does not meet Z (we say two sets meet if they have a nonempty intersection).

Use Hahn–Banach separation to show that, if $x \in X \setminus Z$ then there is a continuous linear functional on X with $f(x) \neq 0$.

Exercise 3.7. Let $0 < p < 1$ and let $X = L^p[0,1]$: The L^p space on the unit interval with Lebesgue measure. The L^p "norm" is not really a norm in this case, since it fails to satisfy the triangle inequality. Instead, we can make a metric d_p on L^p by

$$d_p(u,v) = \|u-v\|_p^p = \int_0^1 |u-v|^p\,dx.$$

Show that d_p is a metric. *Hint*: It is enough to show $|u+v|^p \le |u|^p + |v^p|$ and then integrate this inequality. Since $|u+v| \le |u|+|v|$, it is sufficient to show the inequality when $u,v \ge 0$. That is, show that $(u+v)^p \le u^p + v^p$ for $u,v \ge 0$. This is an equality for $v = 0$. Differentiate wrt v.

It can be shown that L^p is complete in this metric, and the topology induced by this metric makes L^p into a topological vector space as well. (You are not expected to show this, but you are welcome to do it anyway.)

Show that the subspace Z defined in the previous problem is all of X, and conclude from this that the only continuous linear functional on Z is the zero functional. *Hint*: If $u \in L^p$ with $\|u\|_p = 1$, then for any n divide $[0,1]$ into n subintervals $[t_{j-1},t_j]$ so that $\int_{t_{j-1}}^{t_j} |u|^p\,dt = 1/n$ for each j. Then $u = (u_1+\cdots+u_n)/n$ where $u_j = n\chi_{[t_{j-1},t_j]}u$. Show that $\|u_j\|_p^p = n^{p-1}$, and derive the desired result from this.