75048 Mathematical modeling 1998

Exercise 2

Advice/Solution: 1998–09–10/1998–09–17

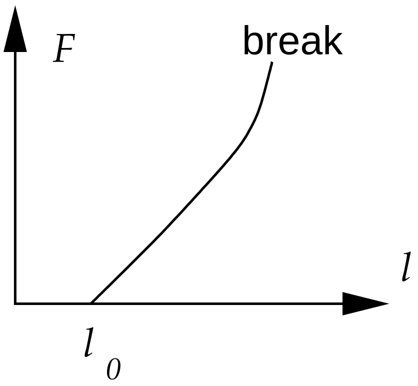
Problem 1: To cook a roast,[1] the roast is placed in an oven and kept there until the center of the roast reaches a predetermined temperature. It is common to model heat transfer from the oven to the roast by stating that the heat flux (thermal energy flow per area and time) is of the form $\beta\cdot(T_1-T)$, where T_1 is the oven temperature and T is the surface temperature of the roast. β is usually known as a heat transfer coefficient. What other physical parameters do you think will influence the time it takes to cook the roast? Write some equations describing heat transfer within the roast as well as a boundary condition involving the heat transfer coefficient, and suitable initial conditions.

Determine all parameters of the problem, and verify that your equations are dimensionally correct. Next, find dimensionless combinations of the parameters, choose suitable scales, and write the equations in dimensionless form. Note that the equations are linear[2] in T, so absolute temperatures are not of interest. You can account for that by letting your rescaled temperature have its zero point at the initial temperature of the roast ($\approx 5\,°\mathrm{C}$, if it came straight out of the refrigerator).

The practically oriented cook is interested in questions like: The recipe says a 2 kg roast should cook for $1\frac{1}{2}$ hours. How long does it take to cook a 3 kg roast?

What advice can you give the cook, if any? How do your uncertainties about the heat transfer coefficient influence the answer? Experience seems to indicate that a high temperature will produce a roast well done on the outside but red in the middle, while a more moderate temperature will produce a roast more evenly done. Discuss this result.

Problem 2 (Exam May 1995): A rubber band will vibrate like a string when stretched. Experiments show that, when the rubber band is stretched to at least twice its initial length, its eigenfrequency is nearly independent of the length. But the frequency rises somewhat just before the band reaches its limit. A series of measurements[3] of length versus required stretching force is given in the graph. Explain the behaviour of the rubber band, based on dimensional analysis and the above facts.



Problem 3: Lin & Segel, Exercise 10 (p. 242).

[1] This exercise can be solved by vegetarians as well: Just imaging cooking a large vegetable; for example, a squash or pumpkin.

[2] Well, heat transfer from oven to roast should really be given by a radiation condition, i.e. proportional to $T_1^4 - T^4$, but let us ignore that for now.

[3] Not really – these “measurements” are faked.

Problem 4 (Exam, January 1986): The equation of motion for a relativistic free electron in an electric field has the form

$$\frac{d}{dt^*}\vec{p}^* = e\vec{E}$$

where[4]

$$\begin{aligned}
\vec{p}^* &= m\vec{v}^*/\sqrt{1-(v^*/c)^2} = \text{momentum of the electron}\\
e &= \text{charge of the electron}\\
\vec{E} &= \text{the electric field}\\
m &= \text{rest mass of the electron}\\
c &= \text{speed of light}\\
\vec{v}^* &= \text{velocity of the electron} (v^* = |\vec{v}^*|)\\
t^* &= \text{time}
\end{aligned}$$

Let now $\vec{E} = E_0\vec{i}$ be a constant field in the x direction, while the electron is given an initial velocity $\vec{v}^*(0) = a\vec{j}$ in the y direction, where $0 < a \ll c$.

a) Show that, with a suitable scaling relevant to the initial movement of the electron, the equation of motion and initial conditions can be written

$$\frac{d}{dt}\left[\frac{\vec{v}}{\sqrt{1-\varepsilon v^2}}\right] = \vec{i}, \quad \vec{v} = \vec{j}, \qquad \text{where } \varepsilon = (a/c)^2.$$

b) Find the first two terms in the perturbation expansion

$$\vec{v}(t) = \vec{v}_0(t) + \varepsilon\vec{v}_1(t) + \varepsilon^2 v_2(t) + \cdots$$

[4]Here we follow the useful convention of adding the asterisk * to *variables* with physical dimensions, while we do not do so for *parameters*.