A very short introduction to bifurcations

Harald Hanche-Olsen
hanche@math.ntnu.no

As an introduction, consider the example of the *damped bead on a rotating hoop*: Consider a bead sliding on a circular hoop made from some wire. Assuming a friction force proportional to the speed of the bead, it satisfies an equation of motion exactly like the circular pendulum: $\ddot{\theta} + \mu\dot{\theta} + \sin\theta = 0$, after scaling.

But now we set the hoop spinning around a vertical axis. If we consider the bead in a coordinate system that rotates with the hoop, we will see two new forces: A centrifugal force and a Coriolis force. The latter will act in a direction perpendicular to the hoop, so we can ignore it (though in practice, it could affect the friction of the bead). The modified equation, with the centrifugal force added, will be of the form

$$\ddot{\theta} + \mu\dot{\theta} + \sin\theta = \Omega^2 \sin\theta\cos\theta$$

where $\Omega \ge 0$ is the (scaled!) angular velocity of the hoop around its vertical axis. This is of course equivalent to the dynamical system

$$\begin{aligned}\dot{\theta} &= \omega,\\ \dot{\omega} &= -\mu\omega - \sin\theta + \Omega^2 \sin\theta\cos\theta.\end{aligned}$$

The equilibrium points of this are easy to find: They are given by

$$\omega = 0 \text{ and } (\sin\theta = 0 \text{ or } \Omega^2\cos\theta = 1).$$

The linearization at an equilibrium is associated with a matrix of partial derivatives of the righthand side of the system:

$$\begin{pmatrix} 0 & 1 \\ -\cos\theta + \Omega^2(\cos^2\theta - \sin^2\theta) & -\mu \end{pmatrix}$$

with the trace $\tau = -\mu$ and the determinant $\delta = \cos\theta - \Omega^2(\cos^2\theta - \sin^2\theta)$. It's slightly more useful to write $\delta = \cos\theta - \Omega^2(2\cos^2\theta - 1)$.

For the bead at the top of the hoop ($\theta = \pi$), we find $\cos\theta = -1$, and therefore $\delta = -(1+\Omega^2) < 0$. Thus this equilibrium point is always a *saddle point*.

The bottom of the hoop ($\theta = 0$) is more interesting. We find $\cos\theta = 1$, so $\delta = 1 - \Omega^2$. If $0 \le \Omega < 1$ then $\delta > 0$, and the equilibrium is *stable* (since $\tau < 0$). But when $\Omega > 1$, the equilibrium is once more a *saddle*, i.e., it is *unstable*.

Finally, let us investigate the equilibrium points given by $\omega = 0$ and $\Omega^2 \cos\theta = 1$. Clearly, we must have $\Omega \ge 1$. With $\cos\theta = \Omega^{-2}$ we get

$$\delta = \Omega^{-2} - \Omega^2(2\Omega^{-4} - 1) = 1 - \Omega^{-2} > 0,$$

so these equilibrium points are also *stable* (again, since $\tau < 0$).

We can sum up our findings in a diagram. Since $\omega = 0$ in all equilibrium points, and the stability (or not) does not depend on μ (so long as $\mu > 0$), we only need to include $\theta$and Ω in the diagram.

A solid curve corresponds to *stable* equilibrium points, wheras a dashed curve corresponds to *unstable* equilibrium points.

The "double point" $(\Omega,\theta) = (1,0)$ on the diagram is a classic *pitchfork* bifurcation. As Ω moves past the critical point, a single stable equilibrium splits, or bifurcates, into a single unstable equilibrium and two stable ones.[1]

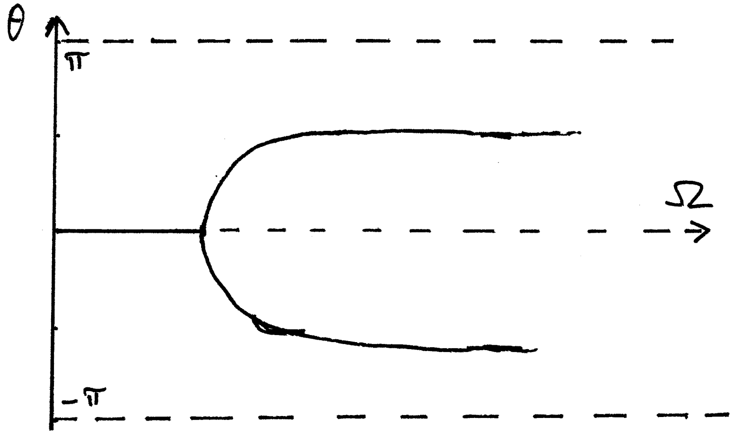

Figure 1: Bifurcation diagram for the bead on a rotating hoop

[1] Since the word "bifurcation" clearly indicates a splitting in *two* parts, maybe the pitchfork should be called a *trifurcation*. But the common usage is to say bifurcation, no matter what.

One dimensional dynamical systems and their bifurcations

A systematic study of two dimensional bifurcations is unfortunately beyond us here, so we limit ourselves to one dimension.

First, we note that the qualitative study of a single differential equation $\dot{x} = f(x)$ is almost embarrasingly simple: Just note that $\dot{x} > 0$ (the system moves to the right) when $f(x) > 0$, whereas $\dot{x} < 0$ (the system moves to the left) when $f(x) < 0$.

So clearly, an equilibrium point ($f(x_0) = 0$) is asymptotically stable is $f(x) > 0$ for x to the left and $f(x) < 0$ to the right of x_0, which is of course the case when $f'(x_0) < 0$.

Similarly, the equilibrium point is unstable is $f(x) < 0$ for x to the left *or* $f(x) > 0$ to the right of x_0, which is of definitely the case (both sides) when $f'(x_0) > 0$.

The conditions on the derivatives are of course just the usual conditions on the eigenvalues of the 1×1 Jacobian matrix of f.

So we now introduce a parameter into our dynamical system, and write it as

$$\dot{x} = f(x, \mu)$$

where the unknown x is a scalar function, and μ is a scalar parameter.

As a specific example, consider the case

$$\dot{x} = f(x, \mu) = -(\mu - x^2 + x^3)(\mu + 4x).$$

The set of equilibrium points can be plotted in the (μ, x) plane[2] as the union of the two curves $\mu = x^2 - x^3$ and $\mu = -4x$.

To work out the stability, just figure out the sign of f in one region: For example $f(x, \mu) < 0$ if $x \gg 1$ and $u > 0$. Then remember that $f(x, \mu)$ changes sign every time (μ, x) crosses one of the two curves in the diagram. So now we know the sign of f in *every* region of the plane.

It is now easy to deduce the stability properties of any point on a curve where $f = 0$. The bifurcation diagram in figure 2 shows all the most common behaviours of one-dimensional bifurcaton problems.

At $(\mu, x) = (-8, 2)$ and at $(\mu, x) = (4, -1)$ you see *double points*.

At $(\mu, x) = (0, 0)$ you see a *pitchfork*. It is in fact a double point too.

All the other points on the set $f = 0$ are *regular points*.

At $(\mu, x) = (\frac{4}{27}, \frac{2}{3})$ you see a *regular turning point*.

[2]Perversely, it is common the draw the μ axis horizontally and the x axis vertically, even though we write $f(x, \mu)$ and not $f(\mu, x)$.

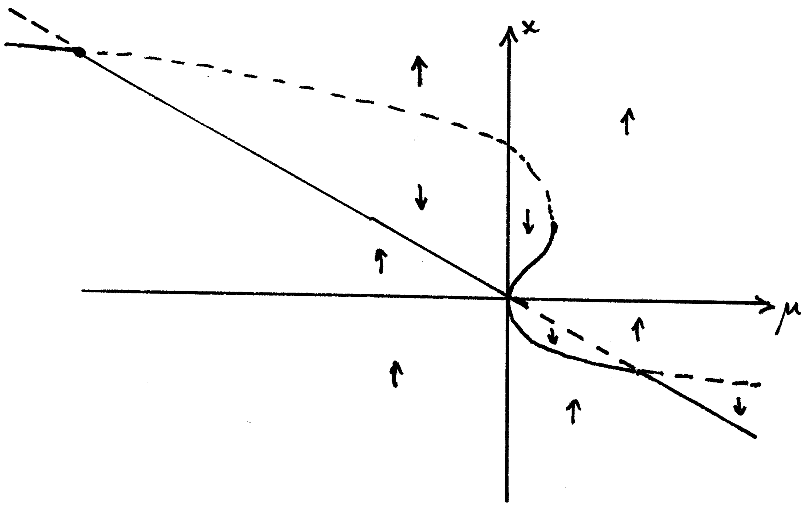

Figure 2: Bifurcation diagram for a one-dimensional system

Let us try to be a bit more systematic. We're assuming that f is sufficiently smooth: In the first parts of the analysis C^1 is enough, but later on we want it to be C^2.

It is a consequence of the implicit function theorem that, near a point on $f = 0$ where $f_\mu \neq 0$ (we write f_x and f_μ for the partial derivatives of f), the set $f = 0$ can be seen as a curve of the form $\mu = g(x)$ in a neighbourhood of the given point. Here g is itself a C^1 function. It satisfies $f\big(x, g(x)\big) = 0$. Differentiating this relation, we get $f_x + f_\mu \cdot g'(x) = 0$, or

$$g'(x) = -\frac{f_x(x,\mu)}{f_\mu(x,\mu)}, \qquad \mu = g(x). \tag{1}$$

Similarly, near points where $f = 0$ and $f_x \neq 0$, we can locally see $f = 0$ as a curve $x = h(\mu)$.

Points on the set $f = 0$ where $f_x \neq 0$ or $f_\mu \neq 0$ are called *regular points.* We have seen that the set $f = 0$ behaves like a curve near any regular point.

A regular point can be a *turning point* if $f_x = 0$ there. From (1) we see that this condition means that the curve $f = 0$ is parallel to the x axis, i.e., vertical the way we have drawn the diagram. All that is required in addition to make it a turning point is a bit of curvature in the curve itself. We could work this out in terms of the partial derivatives of f, but we'll leave that alone.

A *singular* point is one where $f = 0$ and $f_x = f_\mu = 0$ as well. From first year calculus we know that this means we have a *critical point* of f: It could be a local maximum or minimum (if $f_{xx} f_{\mu\mu} - f_{x\mu}^2 > 0$), or it could be a saddle point (if $f_{xx} f_{\mu\mu} - f_{x\mu}^2 < 0$). If $f_{xx} f_{\mu\mu} - f_{x\mu}^2 = 0$ then further analysis is needed.

A local maximum or minimum at a point where $f = 0$ is not very interesting from a dynamical systems perspective.

A saddle point is very different, though: If $f_{xx} f_{\mu\mu} - f_{x\mu}^2 < 0$ at a point where $f = 0$ then we call that point a *double point* of the system.

Near a double point, the set $f = 0$ is the union of two curves that cross each other transversally.[3]

Just like a regular point just might be a turning point, so can a double point possibly be a pitchfork, if one of the two curves happens to be vertical and have a nonzero curvature. Otherwise, the behaviour of a double point is just like we've seen in the example: As the bifurcation parameter μ moves past the critical value, two equilibrium points – one stable and one unstable – approach each other and then cross each other's paths, interchanging their stability properties in the process.

A singular point where $f_{xx} f_{\mu\mu} - f_{x\mu}^2 = 0$ defies any simple general analysis, just like in the study of calculus. In particular cases, however, these cases may not be too hard to work out by hand.

Example: The stirred tank reactor. Imagine substance (a "reactant") dissolved in a liquid. The reactant undergoes a reaction which destroys it (and converts it into something else) at a rate proportional to the reactant concentration c. Many such reactions are strongly temperature dependent. According to a common model (by Arrhenius), the rate is also proportional to $e^{-A/T}$ for some constant A, where T is absolute temperature. Thus, if nothing else happens, c would satisfy a differential equation on the form $\dot{c} = -kce^{-A/T}$.

Many interesting reactions are also *exothermal*, which means they release heat in proportion to the reaction rate. In the above scenario, heat will be released at a rate $hkce^{-A/T}$, so that, if no heat escapes, the temperature will satisfy the eqation $C\dot{T} = hkce^{-A/T}$ (where C is the specific heat capacity of the solvent).

These two equations on their own indicate a reaction that goes slowly to begin with, then speeds up as the solvent heats up. This in turn speeds up the

[3] In this context, *transversally* means "not tangentially".

reaction, which will go faster and faster until most of the reactant has been used up.

The *stirred tank reactor* consists of a tank of a fixed volume V, into which we feed a constant stream of solvent with the reactant in it, and also extract solvent at the same rate, keeping the tank full at all times. Some mechanism is included for stirring the contents, so they are always properly mixed.

If the incoming fluid has temperature T_0 and reactant concentration c_0, and the fluid flows at a rate q, the resulting equations are

$$\begin{aligned} V\dot{c} &= qc_0 - qc - Vkce^{-A/T}, \\ VC\dot{T} &= qCT_0 - qCT + Vhkce^{-A/T}. \end{aligned}$$

We perform a quick'n'dirty nondimensionalization on this, using the time scale (V/q) implied by the mixing process, the concentration scale c_0 (makes excellent sense), and the temperature scale T_0 (makes less sense, but we do it anyway). We also shift the origin of the temperature scale, writing $T = T_0(u+1)$ where u is the dimensionless temperature. The resulting scaled equations (after dropping the primes) are

$$\begin{aligned} \dot{c} &= 1 - c - \frac{c}{\mu}e^{-\alpha/(u+1)}, \\ \dot{u} &= -u + \frac{\beta c}{\mu}e^{-\alpha/(u+1)}. \end{aligned}$$

From this we see that we can get an equation without the common exponential factor: $\dot{u} + \beta\dot{c} = \beta - (u + \beta c)$, which is an ordinary differential equation for $v = u + \beta c$. With this equation, and after replacing βc by $v - u$ in the u equation, we are left with the somewhat simpler system

$$\begin{aligned} \dot{u} &= -u + \frac{v-u}{\mu}e^{-\alpha/(u+1)}, \\ \dot{v} &= \beta - v. \end{aligned}$$

We are presently only interested in investigating equilibrium points and their stability. At an equilibrium, $v = \beta$, and the second equation above certainly implies that this part of the equilibrium is very stable indeed. So for the stability analysis, we can just put $v = \beta$ and work with the single equation

$$\dot{u} = -u + \frac{\beta - u}{\mu}e^{-\alpha/(u+1)}.$$

Any equilibrium of this equation must satisfy

$$\mu u = (\beta - u)e^{-\alpha/(u+1)}.$$

The righthand side of this equation is sketched in figure 3, and the resulting bifurcation diagram is sketched in figure 4. The bifurcation diagram has two regular turning points, with a stable branch at each end and an unstable branch in the middle. During adjustments of the process control parameter (flow rate) μ, the process may “fall off” one stable branch onto the other, resulting in a dramatic change in operating conditions.

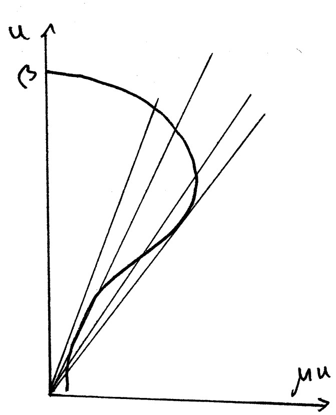

Figure 3: A helpful diagram

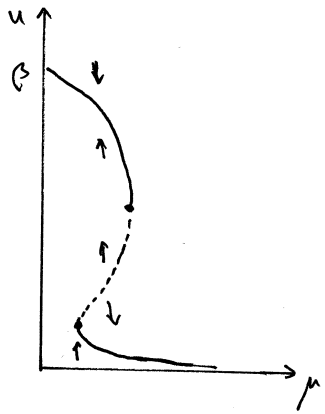

Figure 4: Bifurcation diagram

The above conclusions are not universally valid, however. The lines from the origin in figure 3 correspond to different constant values for μ. The fact that some of them cross the curve in three places, and some just once, corresponds to the fact that u is sometimes triple-valued, and sometimes single-valued, as a function of μ as indicated in figure 4.

One can pick the parameters α and β so that this is not so, however. With a little help from Maple we find that

$$\frac{d^2}{du^2}(\beta - u)e^{-\alpha/(u+1)} = \alpha \frac{\alpha\beta - 2 - 2\beta - (2 + 2\beta + \alpha)u}{(u+1)^4} e^{-\alpha/(u+1)}$$

which is negative for u larger than some critical value, and positive for smaller u. Thus the the curve in figure 3 will have the general convexity properties shown, or else μu will be an everywhere concave function of u for $u \in [0, \beta]$. This is certainly possible, and then the bifurcation diagram will have no turning points, and all equilibrium points are stable.

The Hopf bifurcation

The prototype for a *Hopf bifurcation* is the system

$$\begin{aligned}\dot{x} &= -y + x(\mu - x^2 - y^2),\\ \dot{y} &= \quad x + y(\mu - x^2 - y^2).\end{aligned}$$

It has only one equilibrium point: The origin. (If $\dot{x} = 0$ and $\dot{y} = 0$, multiply the first equation by y and the second by x and subtract.) The linearization at the origin is associated with the matrix

$$\begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}$$

with trace 2μ and determinant $\mu^2 + 1 > 0$: So the origin is stable for $\mu > 0$ and unstable for $\mu < 0$. In fact, the characteristic equation is $\lambda^2 - 2\mu\lambda + \mu^2 + 1 = 0$, with roots $\lambda = \mu \pm i$.

But this change of stability is not the *only* thing that happens when μ passes through 0.

If we introduce polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ then this system becomes

$$\dot{r} = r(\mu - r^2), \quad \dot{\theta} = 1.$$

We see that r has its own little dynamical system which undergoes a pitchfork bifurcation as μ passes through 0. The single stable equilibrium at $r = 0$ (when $\mu < 0$) becomes an unstable equilibrium at $r = 0$ and two stable equilibria at $r = \pm\sqrt{\mu}$ when $\mu > 0$ (but of course, we should ignore the negative r).

That means the full system has a stable limit cycle at $x^2 + y^2 = \mu$ when $\mu >$ 0. The general Hopf bifurcation shows this behaviour: A stable equilibrium becomes an unstable equilibrium surrounded by a stable cycle. (Or the other way around, with the words "stable" and "unstable" interchanged throughout.)

The two complex conjugate roots passing through the imaginary axis while staying away from the linear axis is a typical characteristic of the Hopf bifurcation. But it is not sufficient: The van der Pol system shares this characteristic, but its behaviour as ε passes through zero is more complicated than a Hopf bifurcation.