Some notes on the thermistor

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The steady state equations for the one-dimensional thermistor can be written, on dimensionless form,

$$\frac{\mathrm{d}}{\mathrm{d}z}\Big(\sigma\frac{\mathrm{d}\phi}{\mathrm{d}z}\Big)=0 \qquad \frac{\mathrm{d}^2u}{\mathrm{d}z^2}+\gamma\sigma\Big|\frac{\mathrm{d}\phi}{\mathrm{d}z}\Big|^2=0$$
$$\phi(0)=0 \qquad -\frac{\mathrm{d}u}{\mathrm{d}z}+\beta u=0,\qquad z=0$$
$$\phi(1)=1 \qquad \frac{\mathrm{d}u}{\mathrm{d}z}+\beta u=0,\qquad z=1$$

where the conductivity σ is a function of the temperature u, while β and γ are constants. The equations for ϕ can immediately be integrated and the boundary conditions applied to yield

$$\phi(z)=\frac{\displaystyle\int_0^z\frac{ds}{\sigma(u(s))}}{\displaystyle\int_0^1\frac{dz}{\sigma(u(z))}},\quad\text{and so}\quad \frac{\mathrm{d}\phi}{\mathrm{d}z}=\frac{\dfrac{1}{\sigma(u(s))}}{\displaystyle\int_0^1\frac{dz}{\sigma(u(z))}},$$

which we can then substitute in the equations for u.

Now we add the new assumption $\sigma(u)=e^{-f(u)/\varepsilon}$, where f is a "well behaved" and increasing function, in the sense that $0<f'(z)=O(1)$.[1]

Thus we are left with having to solve this system:

(1)
$$\frac{\mathrm{d}^2u}{\mathrm{d}z^2}+\frac{\gamma e^{f(u)/\varepsilon}}{\Big(\displaystyle\int_0^1 e^{f(u)/\varepsilon}\,\mathrm{d}z\Big)^2}=0$$
$$-\frac{\mathrm{d}u}{\mathrm{d}z}+\beta u=0,\quad z=0\qquad \frac{\mathrm{d}u}{\mathrm{d}z}+\beta u=0,\quad z=1.$$

Before we do anything, note that (1) is unchanged by the substitution $z\curvearrowright 1-z$, so that the solution u will be symmetric about the $\frac12$: $u(1-z)=u(z)$. Also, u is concave, and so it must have a maximum at the center: $u^*=u(\frac12)$.

[1] We should also have $f(0)=0$, since $\sigma(0)=1$ in our scaling.

To proceed further, note that when u changes by ε, then so should $f(u)$, very roughly, in the interesting parts of the temperature range. And so $e^{f(u)/\varepsilon}$ can be expected to change by a factor e. This leads to the suspicion that the maximum of $e^{f(u)/\varepsilon}$ at $z=\frac12$ is very narrow; in fact, ε itself seems like a good candidate for the width of the peak. Thus we are led to consider an *outer region*, where $|z-\frac12|\gg\varepsilon$, and an *inner region*, where $z-\frac12=O(\varepsilon)$.

In the outer region, we should have $u''\sim 0$, so u is approximately linear – but a different linear function for the two halves of the thermistor. Considering the boundary condtions, we are led to

$$u(z)\sim\begin{cases}(1+\beta z)u_0, & 0\le z<\frac12,\\ (1+\beta-\beta z)u_0, & \frac12<z\le 1\end{cases}$$

In the inner region, we might wish to use the scaling $u(z)=u^*-\varepsilon w(\zeta)$ where $z=\frac12+\varepsilon\zeta$. (I chose the minus sign to make w positive.) But the calculations become a bit easier if we use

$$u(z)=u^*-\frac{\varepsilon}{f'(u^*)}w(\zeta),\quad z=\frac12+\varepsilon\zeta$$

instead, since then we get the simpler formulas[2]

$$f(u)\sim f(u^*)-\varepsilon w,\quad e^{f(u)/\varepsilon}\sim\lambda^*e^{-w}\quad(\lambda^*=e^{f(u^*)/\varepsilon}).$$

We plug all this into (1) and get

(2)
$$-\frac{\mathrm{d}^2w}{\mathrm{d}\zeta^2}+\frac{\gamma f'(u^*)}{\varepsilon\lambda^*}\frac{e^{-w}}{\Big(\displaystyle\int_{-\infty}^{\infty}e^{-w}\,\mathrm{d}\zeta\Big)^2}=0$$

At this point, the books says we should choose u^* so that $\gamma f'(u^*)/(\varepsilon\lambda^*)$ becomes 1. But why does it say that, and why the cryptic comment in the margin, that we could use another $O(1)$ constant?

One way to understand this would be to integrate (2) from $-\infty$ to ∞. If we assume that $w'(\zeta)$ has a limit as $\zeta\to\pm\infty$, then this yields the equation

(3)
$$\frac{\gamma f'(u^*)}{\varepsilon\lambda^*}=\Big[\frac{\mathrm{d}w}{\mathrm{d}\zeta}\Big]_{\zeta=-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-w}\,\mathrm{d}\zeta.$$

[2] λ^* is the inverse of the smallest conductivity, which is not quite the same thing as the book says on p. 237.

And so, if our scaling is such that the factors in the righthand side of this equation are $O(1)$, then so is the lefthand side as well.

For more detailed knowledge, however, we should try to *solve* (2). It is after all an equation of the form

$$-w'' + 2c^2 e^{-w} = 0,$$

where the constant c happens to depend globally on w via the integral in (2). So after solving the differential equation, we need to go back and evaluate the integral and fit the result with the original equation.

First we multiply by $-2w'$ and integrate, to get $(w')^2 + 4c^2 e^{-w} = \text{constant}$. Then we use the known data $w(0) = 0$ and $w'(0) = 0$ (because of the symmetry) to evaluate the constant, so we end up with

$$(w')^2 = 4c^2(1 - e^{-w}).$$

Multiply this by $\frac{1}{4}e^w$ to get $(\frac{1}{2}e^{w/2}w')^2 = c^2(e^w - 1)$. With $\psi = e^{w/2}$, this is

$$(\psi')^2 = c^2(\psi^2 - 1).$$

The known data from above yields $\psi(0) = 1$, $\psi'(0) = 0$. The solution then is $\psi(\zeta) = \cosh c\zeta$, so that

$$w = 2\ln\cosh c\zeta.$$

We can now go back and evaluate the problematic integral in (2). Fortunately, this is easier than one might expect, since we know the differential equation satisfied by w, so

$$\int_{-\infty}^{\infty} e^{-w}\,d\zeta = \frac{1}{2c^2}\int_{-\infty}^{\infty} w''\,d\zeta = \frac{1}{2c^2}\Big[w'\Big]_{\zeta=-\infty}^{\infty} = \frac{2}{c} \quad \text{since}$$
$$\Big[w'\Big]_{\zeta=-\infty}^{\infty} = 2c\Big[\tanh c\zeta\Big]_{\zeta=-\infty}^{\infty} = 4c.$$

Substitute these into (3) to conclude that

$$\frac{\gamma f'(u^*)}{\varepsilon\lambda^*} = 8,$$

and not 1 at all. But $8 = O(1)$, so the author is still right; but now we know why.

Notice that we still do not know the value of c, nor do we know the integration constant u_0 of the outer solution. We should be able to find these by matching.

We write up our two term inner solution, and then change to outer variables:

$$U = u^* - \frac{2\varepsilon}{f'(u^*)}\ln\cosh c\zeta = u^* - \frac{2\varepsilon}{f'(u^*)}\ln\cosh\frac{c(z-\frac{1}{2})}{\varepsilon} \tag{4}$$

But $\cosh Z \sim \frac{1}{2}e^{|Z|}$ with a trancendentally small error when $|Z| \to \infty$; therefore $\ln\cosh Z \sim -\ln 2 + |Z|$, and

$$U \sim u^* - \frac{2c}{f'(u^*)}|z - \tfrac{1}{2}|$$

to lowest order. Matching this for $0 \le z < \frac{1}{2}$ with the outer solution $(1+\beta z)u_0$, we get

$$u^* - \frac{c}{f'(u^*)} = u_0, \quad \frac{2c}{f'(u^*)} = \beta u_0,$$

with the solutions

$$u_0 = \frac{2u^*}{2+\beta}, \quad c = \frac{\beta u^* f'(u^*)}{2+\beta}.$$

It is worth noting that we found the width of the center region, where conductivity is low, to be not quite ε, but rather ε/c, as is clear from (4). Therefore, our analysis is not quite right if β is small. This makes sense: If the thermistor is well insulated, it will have a more uniform temperature.