An Introduction to the Mathematical Theory of Nonlinear Control Systems

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Control Systems

$$\dot{x} = f(x,u) \qquad\qquad x \in I\!R^n, \quad u \in U \subset I\!R^m \tag{1}$$

$$x(0) = x_0 \tag{2}$$

A **trajectory** of the system is an *absolutely continuous* map $t \mapsto x(t)$ such that there exist a *measurable* control function $t \mapsto u(t) \in U$, such that $\dot{x}(t) = f\big(x(t), u(t)\big)$ for a.e. $t \in [0,T]$.

Basic assumptions

1 - Sublinear growth

$$\big|f(x,u)\big| \leq C\,\big(1 + |x|\big) \qquad\qquad u \in U, \quad x \in I\!R^n$$

guarantees that solutions remain uniformly bounded, for $t \in [0,T]$.

2 - Lipschitz continuity

$$\big|f(x,u) - f(y,u)\big| \leq L\,|x-y| \qquad\qquad u \in U, \quad x,y \in I\!R^n$$

if $x(\cdot)$, $y(\cdot)$ are trajectories corresponding to the same control function $u(\cdot)$, this implies

$$\big|x(t) - y(t)\big| \leq e^{Lt}\big|x(0) - y(0)\big|$$

hence the Cauchy problem (1)-(2) has a unique solution.

Equivalent Differential Inclusion

$$\dot{x} \in G(x) \doteq \big\{f(x,u)\,; \quad u \in U\big\} \tag{3}$$

A **trajectory** of (3) is an *absolutely continuous* map $t \mapsto x(t)$ such that $\dot{x}(t) \in G\big(x(t)\big)$ for almost every $t \in [0,T]$.

Optimization problems

Select a control function $u(\cdot)$ that performs an assigned task *optimally*, w.r.t. a given cost criterion.

Open-loop control

$u = u(t)$ is a (possibly discontinuous) function of time. Yields an O.D.E. of the form

$$\dot{x}(t) = g(x,t) \doteq f\big(x,\ u(t)\big)$$

where g is Lipschitz continuous in x, measurable in t.
For solutions of the O.D.E., standard existence and uniqueness results hold, for Caratheodory solutions

Closed-loop (feedback) control

$u = u(x)$ is a (possibly discontinuous) function of space. Yields an O.D.E. of the form

$$\dot{x}(t) = g(x) \doteq f\big(x,\ u(x)\big)$$

where g is measurable, possibly discontinuous.
For solutions of the O.D.E., no general existence and uniqueness result is available

1 - Navigation Problem

$x(t) =$ position of a boat on a river

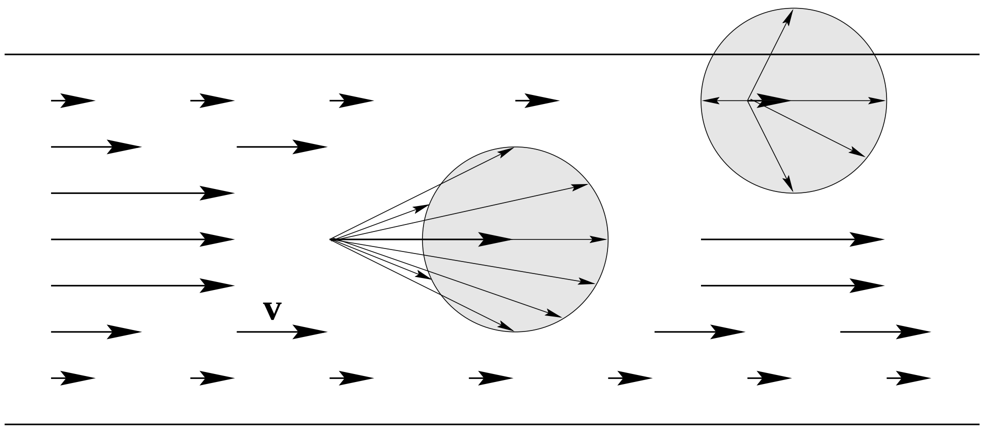
$\mathbf{v}(x)$ velocity of the water

Control system

$$\dot{x} = f(x, u) = \mathbf{v}(x) + \rho u \qquad\qquad |u| \leq 1$$

Differential inclusion

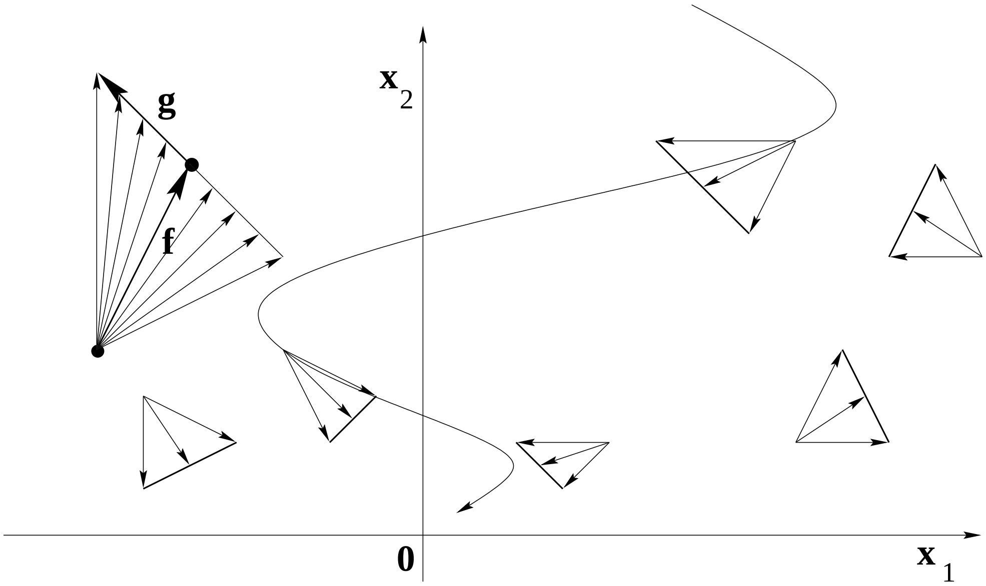
$$\dot{x} \in F(x) = \left\{ \mathbf{v}(x) + \rho u \; ; \qquad |u| \leq 1 \right\}$$



2 - Systems with scalar control entering linearly

$$\dot{x} = f(x) + g(x)\,u \qquad\qquad u \in [-1, 1]$$

$$\dot{x} \in F(x) = \big\{ f(x) + g(x)\,u\ ; \qquad u \in [-1, 1] \big\}$$



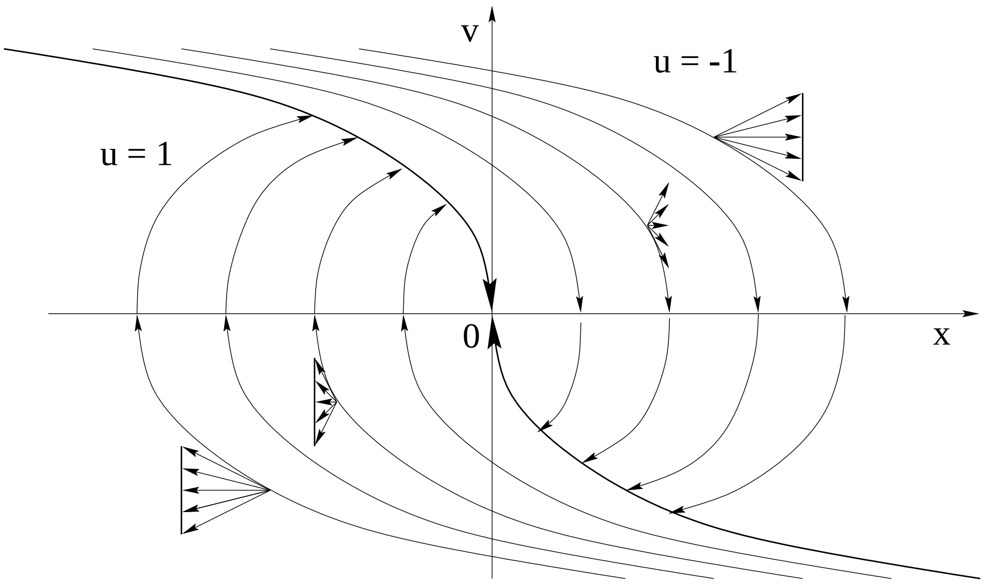
Stabilization in minimum time: Find a control $u(\cdot)$ that steers the system to the origin in minimum time.

Example 2

$$\ddot{x} = u \qquad\qquad u \in [-1, 1]$$

$$\begin{cases} \dot{x} = v \\ \dot{v} = u \end{cases} \qquad\qquad \begin{cases} x(0) = x_0 \\ v(0) = v_0 \end{cases}$$

Find feedback control $u = U(x, v)$ steering the system to the origin in minimum time.

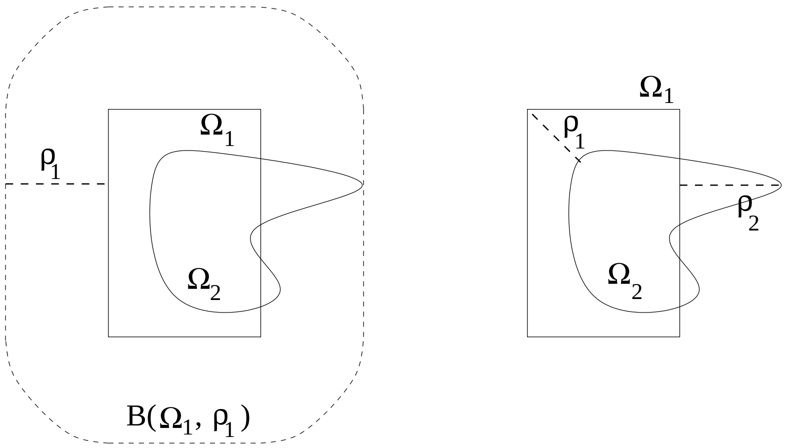


Regularity of Multifunctions

The **Hausdorff distance** between two compact sets $\Omega_1, \Omega_2 \subset I\!R^n$ is

$$d_H(\Omega_1,\Omega_2) \doteq \min\big\{\rho : \quad \Omega_1 \subseteq B(\Omega_2,\rho) \ \text{ and } \ \Omega_2 \subseteq B(\Omega_1,\rho)\big\}$$

$$B(\Omega,\rho) \doteq \ \text{neighborhood of radius } \rho \text{ around the set } \Omega$$



$$d_H(\Omega_1,\Omega_2) = \max\{\rho_1,\rho_2\}$$

$\rho_1 =$ maximum distance of points $p \in \Omega_1$ from Ω_2
$\rho_2 =$ maximum distance of points $p \in \Omega_2$ from Ω_1

A multifunction $x \mapsto G(x) \subset I\!R^n$ is *Lipschitz continuous* if

$$d_H\big(G(x),\ G(y)\big) \leq L\,|x-y| \qquad\qquad x,y \in I\!R^n.$$

If $U \subset I\!R^m$ is compact and f is Lipschitz continuous, then the multifunction

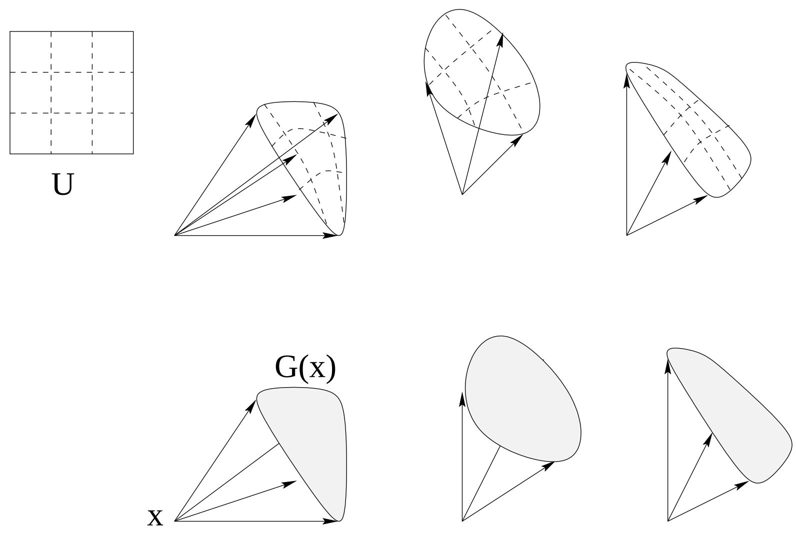
$$x \mapsto G(x) \doteq \big\{f(x,u)\,; \quad u \in U\big\}$$

is Lipschitz continuous

Equivalence with Differential Inclusions

$$\textbf{Control System:} \qquad \dot{x} = f(x,u) \qquad u \in U \tag{1}$$

$$\textbf{Differential Inclusion:} \qquad \dot{x} \in G(x) \tag{2}$$



Theorem 1 (A. Filippov). If f is continuous and U is compact, the set of (Caratheodory) trajectories of (1) coincides with the set of trajectories of (2), with

$$G(x) \doteq \{f(x,u)\,; \quad u \in U\}$$

If $\dot{x}(t) \in G\big(x(t)\big)$ a.e., for each fixed t there exists $u(t) \in U$ such that $\dot{x}(t) = f\big(x(t), u(t)\big)$. The map $t \mapsto u(t)$ can be chosen to be *measurable*.

Theorem 2 (A. Ornelas). Let $x \mapsto G(x)$ be a bounded, Lipschitz continuous multifunction with convex, compact values. Then there exists a Lipschitz continuous function $f : \mathbb{R}^n \times U \mapsto \mathbb{R}^n$ such that $G(x) \doteq \{f(x,u)\,; \quad u \in U\}$ for all x. Here the U (the set of control values) is the closed unit ball in $\mathbb{R}^n$.

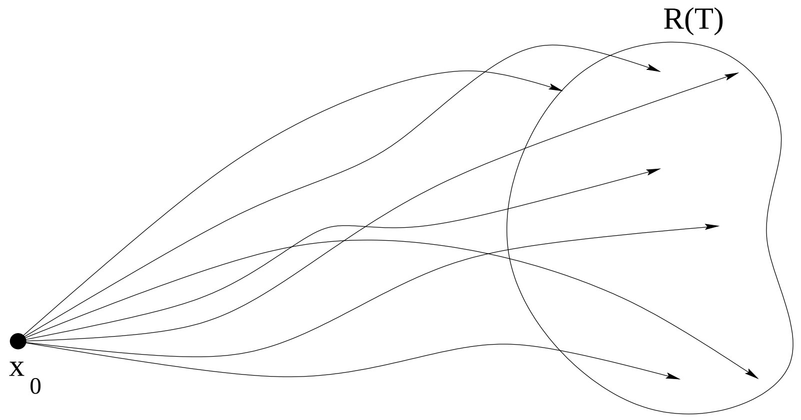
Dynamics of a Control System

- Describe the set of all trajectories of the control system

$$\dot{x} = f(x, u) \qquad u \in U, \qquad x(0) = x_0 \tag{1}$$

- Describe the **Reachable set at time T:**

$$R(T) \doteq \left\{ x(T)\,; \quad x(\cdot) \text{ is a trajectory of (1)} \right\}$$



Ideal case: an explicit formula for $R(T)$

More realistic goal: derive properties of the reachable set

- A priori bounds: $R(T) \subset B(x_0,\ \rho)$
- Is $R(T)$ closed, connected, with non-empty interior ?
- (Local controllability) Is $R(T)$ a neighborhood of x_0, for all $T > 0$?
- (Global controllability) Does every point $x \in \mathbb{R}^n$ lie in the reachable set $R(T)$, for T suitably large?

Linear Systems

$$\dot{x} = Ax + Bu, \qquad\qquad x(0) = 0 \tag{4}$$

A is a $n \times n$ matrix, $\quad B$ is $n \times m$, $\quad u \in \mathbb{R}^m$.

$$x(T) = \int_0^T e^{(T-s)A} Bu(s)\, ds \tag{5}$$

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$$

Theorem 3. For every $T > 0$, the reachable set is

$$R(T) = \mathrm{span}\big\{B,\ AB, A^2B, \cdots, A^{n-1}B\big\}$$

Proof. $R(T)$ is clearly a vector space

$$\mathrm{range}\,\big(e^{tA}\big) \subseteq \mathrm{span}\{I, A, A^2, \ldots\} = \mathrm{span}\big\{I, A, A^2, \ldots A^{n-1}\big\}$$

Hence, by (5), $R(T) \subseteq \mathrm{span}\big\{B,\ AB, A^2B, \cdots, A^{n-1}B\big\}$.

Viceversa, $p \in R(T)^\perp$ implies

$$\Big\langle p\,,\ \frac{dx^k(t)}{dt}\Big\rangle = 0 \qquad\qquad \text{for all } k,\ t \geq 0.$$

At time $t = 0$, taking a constant control $u(t) \equiv \omega$, one finds

$$\big\langle p,\ A^k B\omega\big\rangle = 0 \qquad\qquad \text{for all } \ k \geq 0, \qquad \omega \in \mathbb{R}^m$$

$$p \in \mathrm{span}\{B, AB, A^2B, \ldots A^{n-1}B\}^\perp$$

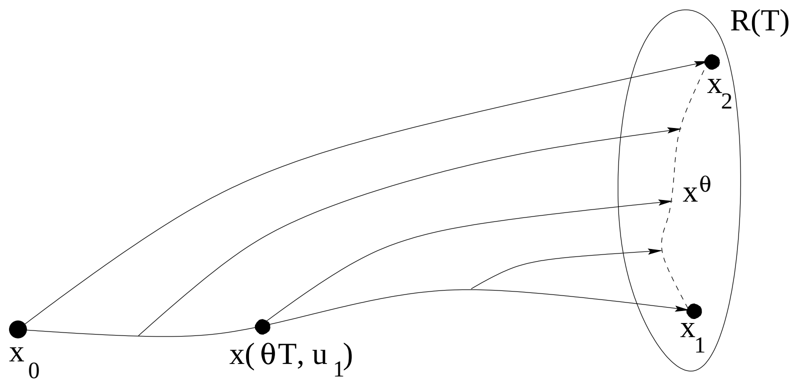
Connectedness

The reachable set $R(T)$ is always connected

$$x_1 = x(T, u_1), \qquad x_2 = x(T, u_2)$$

$$u^\theta(t) \doteq \begin{cases} u_1(t) & \text{if} \quad t \in [0, \theta T] \\ u_2(t) & \text{if} \quad t \in]\theta T,\ T] \end{cases}$$

$\theta \mapsto x(T,\ u^\theta)$ is a path inside $R(T)$, connecting x_1 with x_2



Closure

Example 3. The set of trajectories may not be closed.

$$\dot{x} = u \qquad\qquad u \in \{-1, 1\}, \qquad\qquad x(0) = 0$$

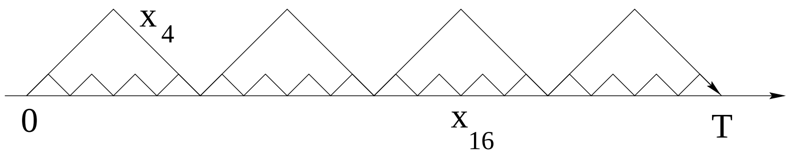
For each $k \geq 1$, define the control $u : [0, T] \mapsto \{-1, 1\}$

$$u_n(t) = \begin{cases} 1 & \text{if} \quad t \in \big[2jT/2n,\ (2j+1)T/2n\big] \\ -1 & \text{if} \quad t \in \big[(2j-1)T/2n,\ 2jT/2n\big] \end{cases} \tag{6}$$

The trajectories $t \mapsto x_n(t)$ satisfy

$$x_n(t) \to x_\infty(t) \equiv 0 \qquad \text{uniformly on } [0, T]$$

but $x_\infty(t) \equiv 0$ is not a trajectory of the system.



Example 4. The reachable set $R(T) \subset \mathbb{R}^2$ may not be closed.

$$(\dot{x}_1, \dot{x}_2) = (u, x_1^2) \qquad u \in \{-1, 1\}, \qquad (x_1, x_2)(0) = (0, 0) \tag{7}$$

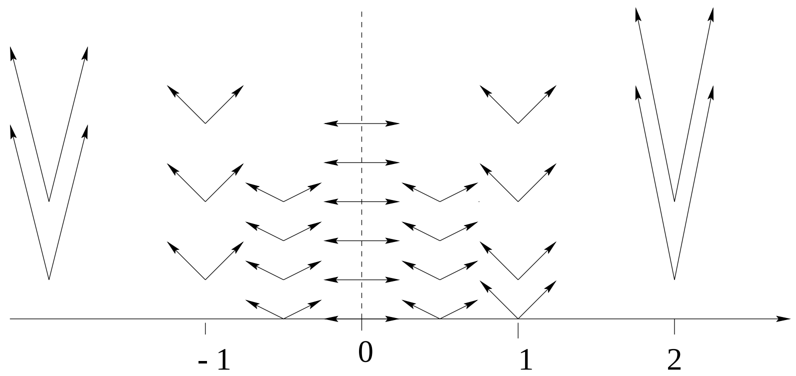
Choosing the controls $u_n(\cdot)$ in (6), at time T we reach the points

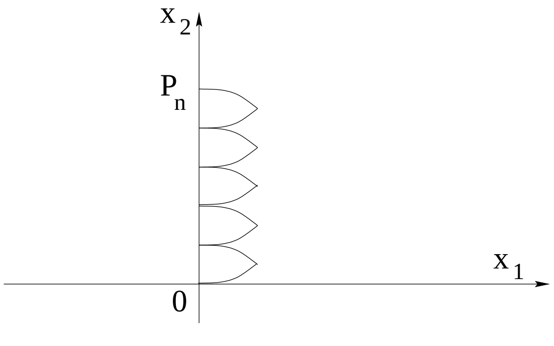
$$P_n = \big(0,\ T^3/12n^2\big)$$

$P_n \to (0,0)$ as $n \to \infty$, but $(0,0) \notin R(T)$. Indeed, for any trajectory of the system

$$x_2(T) = \int_0^T x_1^2(s)\, ds = 0$$

only if $x_1(t) \equiv 0$ for all t, hence $\dot{x}_1(t) = u(t) = 0$ for almost every t. Against the assumption $u(t) \in \{-1, 1\}$





Theorem 4 (A. Filippov). Let $x \mapsto G(x)$ be a Lipschitz continuous multifunction with compact, *convex* values. Then, for every $T \geq 0$ the set of solutions of

$$\dot{x} \in G(x), \qquad\qquad x(0) = x_0 \qquad t \in [0, T]$$

is closed w.r.t. uniform convergence.

Corollary. Under the previous assumptions, the set of trajectories is a compact subset of $\mathcal{C}\big([0,T]\big)$. Moreover, the reachable set $R(T)$ is compact.

Proof. Consider a sequence of trajectories

$$\dot{x}_n(t) \in G\big(x_n(t)\big) \qquad\qquad t \in [0, T]$$

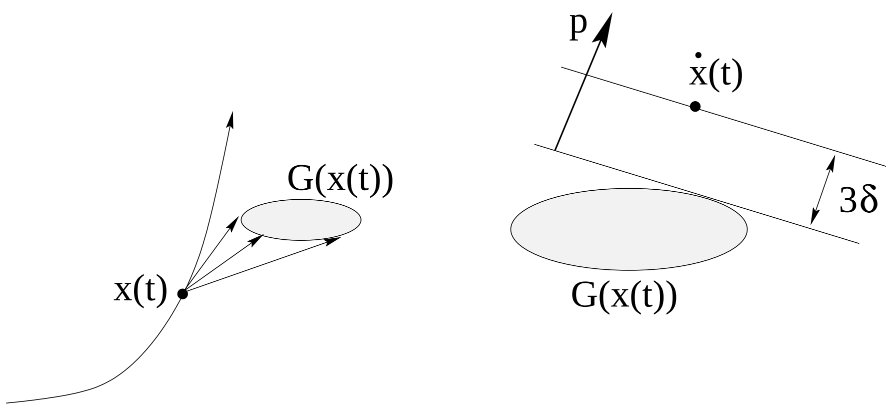
$$x_n(t) \to x(t) \qquad\qquad \text{uniformly on } \; [0, T]$$

Clearly $x(\cdot)$ is Lipschitz continuous, hence differentiable a.e.

Assume $\dot{x}(t) \notin G\big(x(t)\big)$ at some time t. Then $\dot{x}(t)$ is separated by a hyperplane from the convex set $G\big(x(t)\big)$

$$\big\langle p\,,\ \dot{x}(t)\big\rangle \geq 3\delta + \max_{\omega \in G(x(t))} \big\langle p\,,\,\omega\big\rangle$$

$$\Big\langle p,\ \frac{x(t+\varepsilon t) - x(t)}{\varepsilon}\Big\rangle \geq 2\delta + \max_{\omega \in G(x(t))} \big\langle p\,,\,\omega\big\rangle \tag{8}$$



On the other hand, by continuity of G,

$$\max_{\omega \in G(y)} \big\langle p\,,\,\omega\big\rangle \leq \delta + \max_{\omega \in G(x(t))} \big\langle p\,,\,\omega\big\rangle$$

for $\big|y - x(t)\big| \leq \rho$ small. Hence

$$\Big\langle p,\ \frac{x_n(t+\varepsilon t) - x_n(t)}{\varepsilon}\Big\rangle \leq \max_{\omega \in G(y),\ |y-x(t)|\leq\rho} \big\langle p\,,\,\omega\big\rangle \leq \delta + \max_{\omega \in G(x(t))} \big\langle p\,,\,\omega\big\rangle$$

Letting $n \to \infty$, we obtain

$$\Big\langle p,\ \frac{x(t+\varepsilon t) - x(t)}{\varepsilon}\Big\rangle \leq \delta + \max_{\omega \in G(x(t))} \big\langle p\,,\,\omega\big\rangle \tag{9}$$

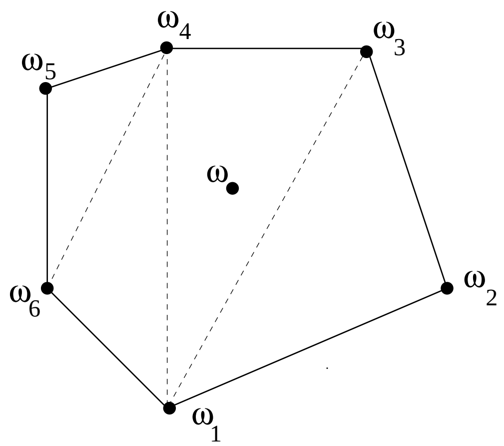
in contradiction with (8).

Convex Closure and Extreme Points

Let $\Omega \subset I\!R^n$ be any compact set
The **convex closure** of Ω, written $\overline{co}\Omega$, is the smallest closed convex set which contains Ω. It admits the representation

$$\overline{\mathrm{co}}\,\Omega = \left\{ \sum_{i=0}^{n} \theta_i \omega_i \qquad \omega_i \in \Omega, \quad \theta_i \geq 0, \quad \sum \theta_i = 1 \right\}$$

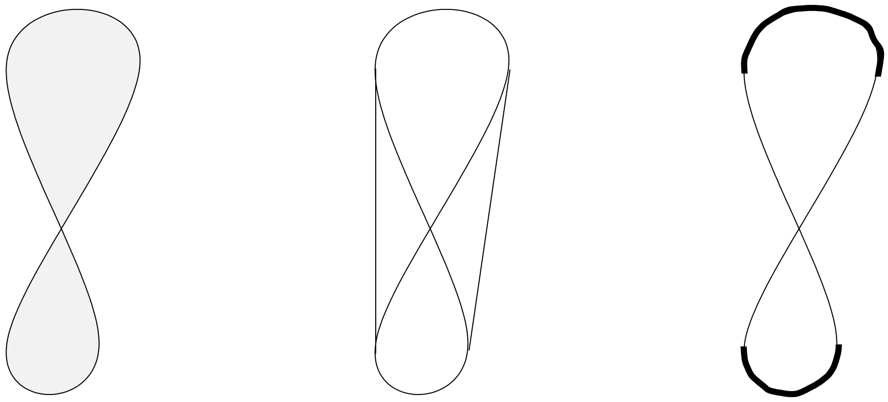
Hence, if $\Omega \subset I\!R^n$, the convex closure $\overline{\mathrm{co}}\,\Omega$ coincides with the set of all convex combinations of $n+1$ elements of Ω.



A point $\omega \in \Omega$ is an **extreme point** if it CANNOT be written as a convex combination

$$\omega = \sum_{i=1}^{k} \theta_i \omega_i, \qquad \theta_i \in [0,1], \quad \sum \theta_i = 1, \qquad \omega_i \in \Omega, \quad \omega_i \neq \omega$$

The set of extreme points is written as $\mathrm{ext}\,\Omega$



Ω $\quad\quad$ $\overline{\text{co}}\ \Omega$ $\quad\quad$ ext Ω

$$\Omega \subset I\!R^n \quad \text{compact} \quad \Longrightarrow \quad \text{ext}\,\Omega \;\subseteq\; \Omega \;\subseteq\; \overline{co}\,\Omega \;=\; \overline{co}\big(\text{ext}\,\Omega\big)$$

Density Theorems

Let $x \mapsto G(x)$ be a multifunction with compact values.
We compare the solutions of the differential inclusions

$$\dot{x} \in \text{ext}\, G(x) \tag{10}$$

$$\dot{x} \in G(x) \tag{11}$$

$$\dot{x} \in \overline{co}\, G(x) \tag{12}$$

Theorem 5 (Relaxation). Let the multifunction $x \mapsto G(x)$ be *Lipschitz continuous* with compact values. Then set of trajectories of (10) is dense on the set of trajectories of (12)