WICK ROTATIONS AND HOLOMORPHIC RIEMANNIAN GEOMETRY

Geometry and Lie Theory,
Eldar Strøme 70th birthday

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REFERENCES

- arXiv:1504.01244, A Wick-rotatable metric is purely electric, Christer Helleland, SH
- arXiv:1503.07354: On holomorphic Riemannian geometry and submanifolds of Wick-related spaces, by Victor Pessers, Joeri Van der Veken
- MINIMUM VECTORS FOR REAL REDUCTIVE ALGEBRAIC GROUPS

  R. W. RICHARDSON AND P. J. SLODOWY

  J. London Math. Soc. (2) 42 (1990) 409-429

WICK ROTATIONS

- Used in physics
- Wick-rotation: $(t, x, y, z) \mapsto (it, x, y, z)$

$$-dt^2 + dx^2 + dy^2 + dz^2 \mapsto dt^2 + dx^2 + dy^2 + dz^2$$

- Done in momentum space or coordinate space to perform certain integrals in, for example, Quantum Field Theories
- Links Statistical mechanics with Quantum mechanics.

WICK ROTATIONS

- In Quantum Field Theories in curved spaces it is desirable to do a similar Wick rotations, however, now the metric is more general.
- Examples: Near black holes, early universe, near galactic centres, etc.
- However, it is not clear whether such a Wick rotation can be done.
- Exact solutions

WICK ROTATIONS

- **Question 1:** Given a Lorentzian spacetime, is it possible to perform a Wick rotation to turn it into a (real)Riemannian space?
- **Question 2:** What pseudo-Riemannian spaces are related through a Wick-rotation?
- **Question 3:** How do we *define* a Wick rotation for a general pseudo-Riemannian manifold?

HOLOMORPHIC RIEMANNIAN GEOMETRY

Definition 1.1. Let E be a complex vector space with a complex structure $\mathbb{J} : E_{\mathbb{R}} \longrightarrow E_{\mathbb{R}}$. Then real linear subspace $W \subset E_{\mathbb{R}}$ is called totally real if $W \cap \mathbb{J}(W) = 0$. If W is a maximal totally real subspace, then W is called a *real form* of E.

In our case our complex vector space *E* will be the complexification of a real vector space: $E = W_{\mathbb{C}} = W \otimes \mathbb{C}$

Definition 1.2. A *holomorphic inner product* is a complex vector space E equipped with a non-degenerate complex bilinear form g.

HOLOMORPHIC RIEMANNIAN GEOMETRY

For a holomorphic inner product space we can always choose an orthonormal frame and identify $E \cong \mathbb{C}^n$

If $X = (X_1, X_2, ..., X_n), \quad Y = (Y_1, Y_2, ..., Y_n)$

$$g_0(X,Y) = X_1Y_1 + ... + X_nY_n,$$

The orthonormal frame bundle for a holomorphic manifold has therefore the structure group $G = O(n, \mathbb{C})$

HOLOMORPHIC RIEMANNIAN GEOMETRY

Definition 1.3. Given a holomorphic inner product space (E, g). Then if $W \subset E$ is a real linear subspace for which $g\big|_W$ is non-degenerate and real valued, i.e., $g(X, Y) \in \mathbb{R}$, $\forall X, Y \in W$, we will call W a *real slice*.

Example: $(\mathbb{C}^n, g_0)$ with standard basis $\{e_1, e_2, ..., e_n\}$

$$W = \mathbb{R}^n_p := \text{span}\{ie_1, ..., ie_p, e_{p+1}, ..., e_n\},$$

with metric

$$h(X, Y) = -X_1Y_1 - ... - X_pY_p + X_{p+1}Y_{p+1} + ... + X_nY_n,$$

for real X, Y

HOLOMORPHIC RIEMANNIAN GEOMETRY

$$\begin{tikzcd}
(\mathbb{C}^n, g_0) \arrow[rr, "O(n,\mathbb{C})"] & & (\tilde{\mathbb{C}}^n, \tilde{g}_0) \\
(W, h) \arrow[u, "\otimes\mathbb{C}"'] & & (\tilde{W}, \tilde{h}) \arrow[u, "\otimes\mathbb{C}"']
\end{tikzcd}$$

HOLOMORPHIC RIEMANNIAN GEOMETRY

Definition 2.1. Given a complex manifold M with complex Riemannian metric g. If a submanifold $N \subset M$ for any point $p \in N$ we have that T_pN is a real slice of (T_pM, g) (in the sense of Defn. **??**), we will call N a real slice of (M, g).

This allows us to define **Wick-related** spaces:

Definition 2.2. Two pseudo-Riemannian manifolds P and Q are said to be *Wick-related* if there exists a holomorphic Riemannian manifold (M, g) such that P and Q are embedded as real slices of M.

WICK ROTATION

A stronger requirement is **Wick-rotated:**

Definition 2.3 (Wick-rotation)**.** If two Wick-related spaces intersect at a point p in M, then we will use the term *Wick-rotation*: the manifold P can be Wick-rotated to the manifold Q (with respect to the point p).

A yet stronger requirement is **standard Wick rotation:**

Definition 2.4 (Standard Wick-rotation)**.** Let the P and Q be Wick-related spaces having a common point p. Then if the tangent spaces T_pP and T_pQ are imbedded in T_pM so that the corresponding Cartan involutions θ and θ' commute – i.e., $[\theta, \theta'] = 0$ – then we say that the spaces P and Q are related through a *standard Wick-rotation*.

HOLOMORPHIC RIEMANNIAN GEOMETRY

Examples: Semi-simple Lie groups as manifolds with bi-inv metric:

$$g(X,Y)=\lambda B(X,Y)$$

where *B* is the Killing form and $\lambda\in\mathbb{R}\setminus\{0\}$

The complexified group has now has several real forms where the Killing forms are restrictions of the Killing form of the complex Lie group. For example, $O(n,\mathbb{C})$ has real forms

$$O(p,q),\quad p+q=n.$$

The various real forms are therefore Wick-rotated w.r.t. each other

HOLOMORPHIC RIEMANNIAN GEOMETRY

- For a holomorphic metric, the Levi-Civita connection is also holomorphic.
- So is the Lie bracket.
- …as well as the Riemann curvature tensor:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

- For two Wick-related spaces the corresponding Riemann tensors are the restriction of the holomorphic Riemann tensor.

FRAME-BUNDLE PERSPECTIVE

- At the common point *p*, two Wick-rotated spaces have to have Riemann tensors lying in the same $O(n,\mathbb{C})$-orbit. Indeed, if the two Wick-rotated spaces have structure groups $O(p,q)$, $O(\tilde{p},\tilde{q})$ then these two have to be imbedded in the complexified structure group $O(n,\mathbb{C})$.

$$\begin{tikzcd} & O(n,\mathbb{C}) & \\ O(p,q) \arrow[ur] & & O(\tilde{p},\tilde{q}) \arrow[ul] \end{tikzcd}$$

ORBITS OF SEMI-SIMPLE GROUP ACTIONS

- Using orthonormal frames, we can consider, for example, the Riemann tensor as a vector, x, in a real vector space *V*. The Riemann tensor Wick-rotated spaces will then be related through the complex orbit:

$$Gx \subset G^{\mathbb{C}}x \cong G^{\mathbb{C}}\tilde{x} \supset \tilde{G}\tilde{x}$$

where

$$G = O(p,q), \;\; \tilde{G} = O(\tilde{p},\tilde{q}), \;\; G^{\mathbb{C}} = O(n,\mathbb{C})$$

ORBITS OF SEMI-SIMPLE GROUP ACTIONS

- **Question1**: Under what conditions do these real orbits intersect, i.e.,

$$Gx \cap \tilde{G}\tilde{x} \neq \emptyset$$

- **Question 2**: Under what conditions are the orbits unique? For what groups and vectors *x* are there no other real orbits?

- **Question 3:** When can a space be Wick-rotated to a different-signature pseudo-Riemannian space?

ALGEBRAIC CLASSIFICATION

- For any tensor (in Lorentzian case) we can classify it algebraically according to boost weight [CMPP]

  Type I/G $R = (R)_{+2} + (R)_{+1} + (R)_0 + (R)_{-1} + (R)_{-2}$
  Type D $R = (R)_0$
  Type II $R = (R)_0 + (R)_{-1} + (R)_{-2}$
  Type III $R = (R)_{-1} + (R)_{-2}$
  Type N $R = (R)_{-2}$

- For the Weyl tensor in 4 dimensions this is equivalent to the Petrov type

ORBITS OF SEMI-SIMPLE GROUP ACTIONS

- Some known results:

  THM: (Richardson-Slodowy): The real orbits are topologically closed if and only if the complex orbits are closed.

- Corollary: A Lorentzian space of (proper) algebraic type II, III or N cannot be Wick rotated to a Riemannian space.

  Proof: For a Riemannian space, $G=O(n)$, is compact, and hence, all orbits are compact and thus closed. For (proper) Petrov type II, III and N, the orbits are not closed.

ORBITS OF SEMI-SIMPLE GROUP ACTIONS

- Recall that a Cartan involution is a map: $\theta: \ \mathfrak{g} \to \mathfrak{g}, \quad \theta^2 = \mathrm{Id}$ so that the following is an inner product of $\mathfrak{g}$:

$$\langle X, Y \rangle = -B(\theta(X), Y).$$

- This can be extended to any tensor product by requiring the Cartan involution acts tensorially.

- A *minimal* vector is a vector $X \in V$ for which

$$||X|| \leq ||g \cdot X||, \forall g \in G$$

ORBITS OF SEMI-SIMPLE GROUP ACTIONS

- THM (Helleland-Hervik): Assume that two spaces, a Riemannian and Lorentzian, respectively, are related through a standard Wick-rotation. Then the Lorentizan spacetime is purely electric.

- A purely electric spacetime is a spacetime for which there exists a Cartan involution of *O(1,n-1),* so that the Riemann tensor is a positive eigenvalue of the Cartan involution.

- Lemma: For the groups above, the orbits above have non-trivial intersection: $Gx \cap \tilde{G}\tilde{x} = \mathcal{M}(\tilde{G}\tilde{x})$
where $\mathcal{M}(\tilde{G}\tilde{x})$ is the set of minimal vectors of the Lorentz group $\tilde{G} = O(1, n-1)$

ORBITS OF SEMI-SIMPLE GROUP ACTIONS

- Here's another result from my student (Helleland)

Let G be an underlying real Lie group of a complex Lie group which is a real form of a complex Lie group $G^{\mathbb{C}}$. We prove that whenever $G^{\mathbb{C}}$ acts on a complex vector space $V^{\mathbb{C}}$, and the restriction to G induces an action on V a real form of $V^{\mathbb{C}}$, then there is a unique real G-orbit in the complex orbit, i.e:

$$G^{\mathbb{C}}v \cap V = Gv, \ \forall v \in V.$$

SUMMARY

- A mathematical setting of Wick rotations have been presented relating the spaces using a holomorphic Riemannian space. Wick rotated pseudo-Riemannian spaces will be real restrictions of this.
- Question: When do real orbits intersect: $Gx \cap \tilde{G}\tilde{x} \neq \emptyset$
- Question: When can a pseudo-Riemannian space be Wick-rotated to a Euclidean-signature space?

Thanks for your attention!

ELDAR: Happy birthday + 1!