Noncommutative Algebraic Geometry, Topology, and Physics

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The relationship between algebraic geometry, topology, and physics, is well documented, and the field is very popular. I shall, in my talk (do my best to) introduce an extension of the methods used up to now, to include my version of non-commutative algebraic geometry.
The interest would be, to *explain* the relationship between notions like Ghost fields, Chern-Simons classes, Maxwell and Bloch's equations, for electromagnetism, resp. time development of spin structures and entanglement, and Seiberg-Witten's monopole equation. I shall, maybe, just vaguely touch upon the beautiful results in low-dimension algebraic topology, based upon these methods, due to Donaldson and his followers.

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0 Mathematical Models in Physics, an Introduction

0 No 1

If we want to study a natural phenomenon, called **P**, we must in the present scientific situation, describe **P** in some mathematical terms, say as a mathematical object, X, depending upon some parameters, in such a way that the changing aspects of **P** would correspond to altered parameter-values for X. This object would be a *model for* **P** if, moreover, X with any choice of parameter-values, would correspond to some, possibly occurring, aspect of **P**.

Two mathematical objects X(1), and X(2), corresponding to the same aspect of **P**, would be called equivalent, and the set, $\mathcal{P}$, of equivalence classes of the objects **P**, would correspond to (possibly a quotient of) the *moduli space*, **M**, of the models, X. The study of the natural phenomena **P**, and its changing aspects, would then be equivalent to the study of the *structure* of $\mathcal{P}$, and therefore to the study of the geometry of the moduli space **M**.

Time and Dynamics

0 No 2

In particular, the notion of *time* would, in agreement with Aristotle and St. Augustin, correspond to some *metric* on this space.
It turns out that to obtain a complete theoretical framework for studying the phenomenon **P**, or the model X, together with its *dynamics*, we should introduce the notion of *dynamical structure*, defined for the space, **M**.
This is done via the construction of a universal non-commutative *Phase Space*-functor, $Ph(-) : Alg_k \to Alg_k$. It extends to the category of schemes, and its infinite iteration $Ph^\infty(-)$, is outfitted with a universal *Dirac derivation*, $\delta \in Der_k(Ph^\infty(-), Ph^\infty(-))$.
A dynamical structure defined on an associative k-algebra $A \in Alg_k$ is now a δ-stable ideal $\sigma \subset Ph^\infty(A)$, and its quotient $A(\sigma) := Ph^\infty(A)/(\sigma)$, with its Dirac derivation. The structure we are interested in, is the *space* $\mathbf{U} := Ph^\infty(\mathbf{M})/\sigma$, corresponding to an open affine covering by algebras of the type, $A(\sigma)$.

Gauge Groups and Measurements

0 No 3

But now we observe that there may be an action of a Lie algebra $\mathfrak{g}$, on $\mathbf{U}$, such that the dynamics of $\mathcal{P}$, really corresponds to that of the quotient $\mathbf{U}/\mathfrak{g}_0$.

To any *open* subset U, of $\mathbf{U}$, there would be associated a, not necessarily commutative, affine k-algebra, $A(\sigma) := \mathbf{O_U}(U)$, with an action of the Lie algebra $\mathfrak{g}_0$, such that the non-commutative quotient $U/\mathfrak{g}_0$, represented by the system $A(\sigma)/\mathfrak{g}_0$ of simple representations of the algebra, $A(\sigma)$, with a $\mathfrak{g}$-connection. This system contains all the available information about the structure of $\mathbf{U}$, restricted to U. An element of $A(\sigma)$ would be called a *local observable*, and wishing to measure the *value* of an observable, leads to the study of the eigenvectors, and their eigenvalues, of the $\mathfrak{g}_0$-invariant representations of this algebra, which as we shall see, is the same as the representations of the system $A(\sigma)/\mathfrak{g}_0$.

Geometry of Representations

0 No 4

We may show that any (geometric:=finitely generated +++) k-algebra A, may be recovered from its finite dimensional simple representations, and that there is an underlying quasi-affine (commutative) scheme-structure on each component $Simp_n(A)$, parametrizing the simple representations of dimension n. In fact, we have shown the following,
There is a commutative k-algebra $C(n)$ with an open subvariety $U(n) \subseteq Simp_1(C(n))$, an étale covering of $Simp_n(A)$, over which there exists a *versal representation* $\tilde{V} \simeq C(n) \otimes_k V$, a vector bundle of rank n defined on $Simp_1(C(n))$, and a *versal family*, i.e. a morphism of algebras,

$$\tilde{\rho} : A \longrightarrow End_{C(n)}(\tilde{V}) \to End_{U(n)}(\tilde{V}),$$

inducing all isoclasses of simple n-dimensional A-modules.

Dynamics and the Dirac Derivation.

0 No 5

$End_{C(n)}(\tilde{V})$ induces also a bundle, of operators, on the étale covering $U(n)$ of $Simp_n(\mathbf{A})$. Assume given a universal derivation, $\delta \in Der_k(A)$. Pick any $v \in Simp_n(A)$ corresponding to the right A-module V, with structure homomorphism $\rho_v : A \to End_k(V)$, then δ composed with ρ_v, gives us an element,

$$\delta_v \in Ext^1_A(V, V)).$$

Therefore, δ defines a unique one-dimensional distribution in $\Theta_{Simp_n(A)}$, which, once we have fixed a versal family, defines a vector field, $[\delta] \in \Theta_{Simp_n(A)}$, and in good cases, a (rational) derivation, the

$$Dirac\ Derivation : [\delta] \in Der_k(C(n)),$$

governing the dynamics of our model.
Problem: How do we find this universal derivation?

I Phase spaces of associative algebras

I No 1

Given an associative k-algebra A, denote by $A/k - \underline{alg}$ the category where the objects are homomorphisms of k-algebras $\kappa : A \to R$, and the morphisms, $\psi : \kappa \to \kappa'$ are commutative diagrams,

$$\begin{tikzcd} & A \arrow[dl, "\kappa"'] \arrow[dr, "\kappa'"] & \\ R \arrow[rr, "\psi"'] & & R' \end{tikzcd}$$

and consider the functor,

$$Der_k(A, -) : A/k - \underline{alg} \longrightarrow \underline{Sets}.$$

It is representable by a k-algebra-morphism, $\iota : A \longrightarrow Ph(A)$, with a universal family, i.e. a universal derivation,

$$d : A \longrightarrow Ph(A).$$

A universal derivation associated to an A-module

I No 2

Clearly we have the identities,

$$d_* : Der_k(A, A) = Mor_A(Ph(A), A),$$

and,

$$d^* : Der_k(A, Ph(A)) = End_A(Ph(A)),$$

the last one associating d to the identity endomorphism of Ph. Let now V be a right A-module, with structure morphism

$$\rho : A \to End_k(V).$$

We obtain another universal derivation,

$$c : A \longrightarrow Hom_k(V, V \otimes_A Ph(A)),$$

defined by, $c(a)(v) = v \otimes d(a)$.

The Kodaira-Spencer class

I No 3

Using the long exact sequence, of Hochschild cohomology,

$$0 \to Hom_A(V, V \otimes_A Ph(A)) \to Hom_k(V, V \otimes_A Ph(A))$$
$$\xrightarrow{\iota} Der_k(A, Hom_k(V, V \otimes_A Ph(A))) \xrightarrow{\kappa} Ext^1_A(V, V \otimes_A Ph(A)) \to 0,$$

and,

$$c \in Der_k(A, Hom_k(V, V \otimes_A Ph(A))),$$

we obtain the non-commutative *Kodaira-Spencer class*,

$$c(V) := \kappa(c) \in Ext^1_A(V, V \otimes_A Ph(A)),$$

inducing, via the identity d_*, the *Kodaira-Spencer morphism*,

$$g : \Theta_A := Der_k(A, A) \longrightarrow Ext^1_A(V, V).$$

Hamiltonians and Connections

I No 4

Using again the long exact sequence, of Hochschild cohomology,

$$0 \to Hom_A(V,V) \to Hom_k(V,V) \xrightarrow{\iota} Der_k(A, Hom_k(V,V)) \xrightarrow{\kappa} Ext^1_A(V,V) \to 0,$$

we prove,

Theorem

Let $\rho : A \to End_k(V)$, be an A-module, and let $\delta \in Der_k(A, Hom_k(V,V))$, map to 0 in $Ext^1_A(V,V)$, i.e. assume $\kappa(\delta) = 0$, then there exist an element, $Q_\delta \in Hom_k(V,V)$, the Hamiltonian, such that for all $a \in A$,

$$\rho(\delta(a)) = [Q_\delta, \tilde{\rho}(a)].$$

If V is a simple A-module, $ad(Q_\delta)$ is unique.

Chern Characters

I No 5

As is well known, in the commutative case, the Kodaira-Spencer class gives rise to a Chern character by putting,

$$ch^i(V) := 1/i!\ c^i(V) \in Ext^i_A(V, V \otimes_A Ph(A)),$$

and if $c(V) = 0$, the curvature $R(V)$ of ∇, induces a curvature class,

$$R_\nabla \in H^2(k, A; \Theta_A, End_A(V)).$$

The iterated Phase Space functor, Ph^*

I No 7

The phase-space construction may, be iterated. Given the k-algebra A we may form the sequence, $\{Ph^n(A)\}_{0\leq n}$, defined inductively by

$$Ph^0(A) = A,\ Ph^1(A) = Ph(A), ..., Ph^{n+1}(A) := Ph(Ph^n(A)).$$

Let $i_0^n : Ph^n(A) \to Ph^{n+1}(A)$ be the canonical imbedding, and let $d_n : Ph^n(A) \to Ph^{n+1}(A)$ be the corresponding derivation. Since the composition of i_0^n and the derivation d_{n+1} is a derivation $Ph^n(A) \to Ph^{n+2}(A)$, corresponding to the homomorphism,

$$Ph^n(A) \to^{i_0^n} Ph^{n+1}(A) \to^{i_0^{n+1}} Ph^{n+2}(A)$$

there exist by universality a homomorphism $i_1^{n+1} : Ph^{n+1}(A) \to Ph^{n+2}(A)$, such that,

$$i_0^n \circ i_1^{n+1} = i_0^n \circ i_0^{n+1}$$

and such that,

$$d_n \circ i_1^{n+1} = i_0^n \circ d_{n+1}.$$

The Semi-Cosimplicial Structure of Ph*

I No 8

Clearly we may continue this process constructing new homomorphisms,

$$\{i_j^n : Ph^n(A) \to Ph^{n+1}(A)\}_{0 \leq j \leq n},$$

such that,

$$i_p^n \circ i_0^{n+1} = i_0^n \circ i_{p+1}^{n+1}$$

with the property,

$$d_n \circ i_{j+1}^{n+1} = i_j^n \circ d_{n+1}.$$

We find the following identities,

$$\begin{aligned} i_p^n i_q^{n+1} &= i_{q-1}^n i_p^{n+1}, \ p < q \\ i_p^n i_p^{n+1} &= i_p^n i_{p+1}^{n+1} \\ i_p^n i_q^{n+1} &= i_q^n i_{p+1}^{n+1}, \ q < p. \end{aligned}$$

The Semi-Cosimplicial Structure of Ph*

I No 9

To see this, compose with i_0^{n-1} and d_{n-1}, and use induction. Thus, the $Ph^*(A)$ is a semi-co-simplicial k-algebra with a co-section h_0, onto A. And it is easy to see that h_0 together with the corresponding co-sections $h_p : Ph^{p+1}(A) \to Ph^p(A)$, for $Ph^p(A)$ replacing A, form a trivialising homotopy for $Ph^*(A)$. Thus, we have,

$$H^n(Ph^*(A)) = 0, n \geq 0,$$

i.e.

- $Ph^*(A)$ is a cosimplicial resolution of the algebra A.

Therefore, for any object,

$$\kappa : A \to R \in A/k - \underline{alg}$$

the co-simplicial algebra above induces simplicial sets,

$$Mor_k(Ph^\star(A), R),\ Mor_A(Ph^\star(A), R),$$

and one should be interested in the homotopy.

Homotopy

I No 10

See also that this generalizes to a canonical functor,

$$Spec : (k - alg^{\Delta})^{op} \longrightarrow SPr(k)$$

where $(k - alg)^{\Delta}$ is the category of co-simplicial k-algebras, and $SPr(k)$ is the category of simplicial presheves on the category of k-schemes enriched by some Grothendieck topology.
As usual, the imbedding of the category of k-algebras in the category of co-simplicial algebras is defined simply by giving any k-algebra a constant co-simplicial structure. The fact that $Ph^{\star}(A)$ is a resolution of A, is therefore simply saying that,

$$Spec(Ph^{\star}((A)) \rightarrow Spec(A),$$

is a week equivalence in $SPr(k)$.
This might be a starting point for a theory of homotopy for (non-commutative) k-schemes.

Cohomology

I No 11

We may also consider, for any k-algebra R, the simplicial k-vectorspace,

$$Der_k(Ph^\star(A), R),$$

Consider this complex for $R = A$, i.e. $Mor_A(Ph^\star(A), A)$. Clearly, $Mor_A(Ph^{n+1}(A), A) = Der_k(Ph^n(A), A), n \geq 0$, and we have,

$$Mor_A(Ph^n(A), A) = \{\xi_0 \circ \xi_{i_1} \circ \ldots \circ \xi_{i_r} | 0 \leq i_l \leq i_{l+1} \leq n | \xi_0 = id_A, \xi_i \in Der_k(A), i$$

Since Ph is a functor, and Ph^{*+1} is a co-simplicial resolution of A. we may apply this to any scheme X, given in terms of an affine covering $\mathbf{U}$, and obtain an algebraic homology (or cohomology), with converging spectral sequences,

$$E^1_{pq} = H_p(H_{\mathbf{U}}{}^{-q}(Der_k(Ph^\star(A), A))), E_{q,p} = H^{-q}_{\mathbf{U}}(H_p(Der_k(Ph^\star(A), A)))$$

If we, in $Mor_A(Ph^n(A), A)$, identify $\xi \sim \alpha\xi, \alpha \in k^*$, we obtain a rational cohomology with converging spectral sequences,

$$E_1^{pq} = H^p(H^q_{\mathbf{U}}(Mor_A(Ph^n(A), A), \mathbf{Q})), E_2^{q,p} = H^q_{\mathbf{U}}(H^p(Mor_A(Ph^n(A), A), \mathbf{Q}))$$

Relation to the de Rham complex I

I No 12

Consider now the diagram,

$$\begin{tikzcd}
A \arrow[r, "i_0^0"] & Ph(A) \arrow[r, "i_p^1"] & Ph^2(A) \arrow[r, "i_p^2"] & Ph^3(A) \arrow[r, "i_p^3"] & {} \\
 & \mathfrak{m}_1^1 \arrow[u] \arrow[r, "i_p^1"] & \mathfrak{m}_2^1 \arrow[u] \arrow[r, "i_p^2"] & \mathfrak{m}_3^1 \arrow[u] \arrow[r, "i_p^3"] & {}
\end{tikzcd}$$

where, for each integer n, the symbol i_p^n, for $p = 0, 1, ..., n$ signify the family of A-morphisms between $Ph^n(A)$ and $Ph^{n+1}(A)$ defined above, and where $\mathfrak{m}_n^1$ is the ideal of $Ph^n(A)$ generated by $im(d)$, which is the same as the ideal generated by the family, $\{i_p^{n-1}(i_p^{n-2}(...(i_p^1(d(A))...))\}$, for all possible p. And, inductively, let $\mathfrak{m}_n^m$ be the ideal generated by $\mathfrak{m}_n^1\mathfrak{m}_n^{m-1}$.

Relation to the de Rham complex II

I No 13

We find an extended diagram,

$$
\begin{tikzcd}
A \arrow[r, "i_0^0"] \arrow[d] & Ph(A) \arrow[r, "i_p^1"] \arrow[d] & Ph^2(A) \arrow[r, "i_p^2"] \arrow[d] & Ph^3(A) \arrow[r, "i_p^3"] \arrow[d] & \cdots \\
A \arrow[r, "i_0^0"] \arrow[dr, "d"] & A \arrow[r, "i_p^1"] \arrow[dr, "d"] & A \arrow[r, "i_p^2"] \arrow[dr, "d"] & A \arrow[r, "i_p^3"] \arrow[dr, "d"] & \cdots \\
 & \mathfrak{m}_1^1/\mathfrak{m}_1^2 \arrow[r, "i_p^1"] \arrow[dr, "d"] & \mathfrak{m}_2^1/\mathfrak{m}_2^2 \arrow[r, "i_p^2"] \arrow[dr, "d"] & \mathfrak{m}_3^1/\mathfrak{m}_3^2 \arrow[r, "i_p^3"] \arrow[dr, "d"] & \cdots \\
 & \mathfrak{m}_1^2/\mathfrak{m}_1^3 \arrow[r, "i_p^1"] & \mathfrak{m}_2^2/\mathfrak{m}_2^3 \arrow[r, "i_p^2"] & \mathfrak{m}_3^2/\mathfrak{m}_1^3 \arrow[r, "i_p^3"] & \cdots
\end{tikzcd}
$$

The diagonals are not necessarily complexes, but to kill all d^n, $n \geq 2$, it suffices to kill d^2, and for this it suffices to kill $d_1 d_0$, as one easily see, applying the edge homomorphisms to, $d_1(d_0(a))$ for all $a \in A$.

Curvature

I No 14

Definition

The curvature $R(A)$ of the associative k algebra, A, is the k-linear map composition of d_0 and d_1,

$$R(A) = d_0 d_1 : A \to \mathfrak{m}_2^2/\mathfrak{m}_2^3.$$

Now, kill the curvature $R(A)$, and all the terms under the first diagonal, beginning with $\mathfrak{m}_1^2/\mathfrak{m}_1^3$, together with all terms generated by the actions of the edge homomorphisms on these terms, and let, $\mathbf{\Omega}_n^m$ be the quotient of $\mathfrak{m}_n^m/\mathfrak{m}_n^{m+1}$, for $n \geq 0$. Clearly, $\Omega_n^0 = A$ for all $n \geq 0$, and we have got a graded semi co-simplicial A-module, with a k-differential d, such that $d^2 = 0$,

Generalized de Rham complex.

I No 15

The diagram is now looking like,

$$
\begin{tikzcd}
A \arrow[r, "i_0^0"] \arrow[d] & Ph(A) \arrow[r, "i_p^1"] \arrow[d] & Ph^2(A) \arrow[r, "i_p^2"] \arrow[d] & Ph^3(A) \arrow[r, "i_p^3"] \arrow[d] & {} \\
A \arrow[r, "i_0^0"] \arrow[dr, "d"] & A \arrow[r, "i_p^1"] \arrow[dr, "d"] & A \arrow[r, "i_p^2"] \arrow[dr, "d"] & A \arrow[r, "i_p^3"] \arrow[dr, "d"] & {} \\
& \Omega_1^1 \arrow[r, "i_p^1"] \arrow[dr, "d"] & \Omega_2^1 \arrow[r, "i_p^2"] \arrow[dr, "d"] & \Omega_3^1 \arrow[r, "i_p^3"] \arrow[dr, "d"] & {} \\
& & \Omega_2^2 \arrow[r, "i_p^2"] & \Omega_3^2 \arrow[r, "i_p^3"] & {}
\end{tikzcd}
$$

It is therefore a graded complex, in two ways. First as a complex induced from the semi-cosimplicial structure, with differential of bidegree (1,0), and second, as complex with differential d, of bidegree (1,1).

The commutative case

I No 16

Consider now the complex,

$$A \to^d \Omega_1^1 \to^d \Omega_2^2 \to^d \Omega_3^3 \to^d \ldots$$

.

Theorem

Suppose A is commutative, then there is a natural morphism of complexes of A-modules,

$$\Omega_A^\star \subset \Omega_\star^\star,$$

with,

$$\Omega_A^n \simeq \Omega_n^n.$$

The proof

I No 17

Let, $a_i \in A, i = 1, ..., r$, and compute in $\Omega^r_\star$ the value of, $d^r(a_1 a_2 ... a_r)$. It is clear that this gives the formula,

$$\sum d_{i_1}(a_1) d_{i_2}(a_2) .. d_{i_r}(a_r) = 0,$$

the sum being over all permutation $(i_1, i_2, ..., i_r)$ of $(0, 1, ..., r-1)$. Here we consider A as a subalgebra of $Ph^n(A)$ via the unique compositions of the $i^s_0 : Ph^s(A) \subset Ph^{s+1}(A)$. In particular, we have,

$$d_0(a_1) d_1(a_2) + d_1(a_1) d_0(a_2) = 0,$$

for all $a_1, a_2 \in A$. This relation and the relation $d_0(a) d_1(b) = d_1(b) d_0(a)$, which follows from commutativity, $d(a)b = bd(a)$, forcing the left and right A-action on Ω_A to be equal, immediately give us, $d_0(a) d_1(b) = -d_0(b), d_1(a)$. It is now clear that the map that sends the element $da_1 \wedge da_2 \wedge ... \wedge da_r \in \Omega^r_A$ to $d_0(a_1) d_1(a_2) .. d_{r-1}(a_r) \in \Omega^r_r$ is an isomorphism, and the rest should be clear.

Generalization to modules I

I No 18

Let now, V be an A-module, and assume $c(V) = 0$, and pick a connection, $\nabla \in Hom_k(V, V \otimes_A Ph(A))$ with $c = \iota(\nabla)$. This imply that for $a \in A$ and $v \in V$ we have $\nabla(va) = \nabla(v)a + v \otimes d_0(a)$. Composing ∇ with the cosection, $o : Ph(A) \to A$, corresponding to the 0-derivation of A, we therefore obtain an A-linear homomorphism $P : V \to V$, *a potential*. Since $i_0^0 : A \to Ph(A)$ is a section of o, we find a k-linear map,

$$\nabla_0 := \nabla - P : V \to V \otimes \mathfrak{m}_1^1$$

Using the property,

$$d_n \circ i_{j+1}^{n+1} = i_j^n \, d_{n+1},$$

it is easy to find well defined k-linear maps,

$$\nabla_1 : V \to V \otimes \Omega_2^2, \ \nabla_2 : V \to V \otimes \Omega_3^3, ..., \nabla_n : V \to V \otimes \Omega_{n+1}^{n+1} \ \forall n \geq 0,$$

given by ,

$$\nabla_{n+1} := \nabla_n \circ i_1^{n+1}, \ n \geq 0.$$

Generalization to modules

I No 19

Fix the connection ∇. For all, $v \in V, \omega \in \Omega^n_n$, the formula,

$$\nabla_n(v \otimes \omega) = \nabla_n(v)\omega + v \otimes d_n(\omega).$$

makes sense, and defines *derivations*, also called d. We obtain a situation just like above,

$$\begin{tikzcd}
V \arrow[r, "i_0^0"] \arrow[d] & V \otimes Ph(A) \arrow[r, "i_p^1"] \arrow[d] & V \otimes Ph^2(A) \arrow[r, "i_p^2"] \arrow[d] & V \otimes Ph^3(A) \arrow[r, "i_p^3"] \arrow[d] & {} \\
V \arrow[r, "i_0^0"] \arrow[dr, "d"] & V \arrow[r, "i_p^1"] \arrow[dr, "d"] & V \arrow[r, "i_p^2"] \arrow[dr, "d"] & V \arrow[r, "i_p^3"] \arrow[dr, "d"] & {} \\
& V \otimes \Omega^1_1 \arrow[r, "i_p^1"] \arrow[dr, "d"] & V \otimes \Omega^1_2 \arrow[r, "i_p^2"] \arrow[dr, "d"] & V \otimes \Omega^1_3 \arrow[r, "i_p^3"] \arrow[dr, "d"] & {} \\
& & V \otimes \Omega^2_2 \arrow[r, "i_p^2"] & V \otimes \Omega^2_3 \arrow[r, "i_p^3"] & {}
\end{tikzcd}$$

Generalization to modules

I No 20

In general, there are no reasons for these $d's$ to define complexes, and we shall make the following definition,

Definition

The curvature $R(V, \nabla)$ of the connection ∇ defined on the right k A-module V, is the k-linear map composition of d_0 and d_1,

$$R(V, \nabla) = d_0 d_1 : V \to V \otimes \Omega_2^2.$$

The following result is then easily proved,

Theorem

Suppose A is commutative, and let $\nabla : \Theta_A \to End_k(V)$ be the connection corresponding to ∇_0. Suppose moreover that the curvature R of ∇ is 0, then $R(V) = 0$, implying that $d^2 = 0$, and so the diagonals in the diagram above, are all complexes.

Dynamics and the Dirac Derivation

I No 21

Consider now the co-simplicial algebra,

$$A \to^{i_0^0} Ph(A) \to^{i_p^1} Ph^2(A) \to^{i_p^2} Ph^3(A) \to^{i_p^3} ....$$

where, for each integer n, the symbol i_p^n, for $p = 0, 1, ..., n$ signify the family of A-morphisms between $Ph^n(A)$ and $Ph^{n+1}(A)$ defined above. The inductive (direct) limit,

$$Ph^\infty(A) = \varinjlim_{n \geq 0} \{Ph^n(A), i_j^n\}$$

comes with homomorphisms

$$i_n : Ph^n(A) \longrightarrow Ph^\infty(A), \text{satisfying} i_j^n \circ i_{n+1} = i_n, \quad j = 0, 1, \ldots, n$$

Moreover, the family of derivations, $\{d_n\}_{0 \leq n}$ defines a unique Dirac-derivation,

$$\delta : Ph^\infty(A) \longrightarrow Ph^\infty(A),$$

such that $i_n \circ \delta = d_n \circ i_{n+1}$.

Dynamics in Rep(A)

I No 25

Since $Ext^1_A(V,V)$ is the tangent space of the miniversal deformation space of V as an A-module, we see that the non-commutative space $Ph(A)$ also parametrizes the set of generalised momenta, i.e.the set of pairs of an A-module V, and a tangent vector of the formal moduli of V, at that point. Therefore the above implies that any representation, $\rho : Ph^\infty(A) \to End_k(V)$, corresponds to a family of $Ph^n(A)$-module-structures on V, for $n \geq 1$, i.e. to an A-module $\rho_0 : (V_0 := V)$, an element $\eta_0 \in Ext^1_A(V,V)$, i.e. a tangent of the deformation functor of $V_0 := V$, as A-module, an element $\eta_1 \in Ext^1_{Ph(A)}(V,V)$, i.e. a tangent of the deformation functor of $V_1 := V$ as $Ph(A)$-module, an element $\eta_2 \in Ext^1_{Ph^2(A)}(V,V)$, i.e. a tangent of the deformation functor of $V_2 := V$ as $Ph^2(A)$-module, etc.

Dynamics in Rep(A)

I No 26

All this is just $\rho_0 : A \to End_k(V)$, considered as an A-module, together with a sequence $\{\eta_n\}, 0 \leq n$, of a tangent, or a *momentum*, η_0, an acceleration vector, η_1, and any number of higher order *momenta* η_n.

- Thus, specifying a $Ph^{\infty}(A)$-representation V, implies specifying a *formal curve* through $\rho_0(v_0)$, the base-point, of the *miniversal deformation space* of the A-module V. Formally, this curve is given by the composition of the homomorphism $\epsilon(\tau) := exp(\tau\delta)$ and ρ.

Preparation and Time Evolution of Measurements

I No 27

It is, however, impossible to prepare a physical situation such that a measurement, i.e. an object like ρ_0, is given by an infinite sequence $\{\eta_n\}$, of dynamical data. We shall have to be satisfied with a finite number of data, and normally with just the first one, i.e. the momentum η_0. This is the Problem of Preparation and (of the) Time Evolution of a representation ρ, to be treated in the sequel.

Finite Dimensional Representations of $Ph^{\infty}(k[\underline{t}])$

I No 28

Theorem

Given an r-dimensional $k[\underline{t}] := k[t_1, ..., t_d]$*-module, consisting of* r *points* $\{P_p = (\alpha_1^0(p), \alpha_2^0(p), ..., \alpha_d^0(p))\}_{p=1,...,r}$. *Assume given,* $\alpha_i^n(p) \in k,\ i = 1, ..., d,\ n \geq 0,\ p = 1, ..., r$, *and arbitrary coupling constants,* $\sigma_m(p, q) \in k$ *with* $\sigma_0 = 0$. *Put* $\alpha_i^n(p, q) = \alpha_i^n(p) - \alpha_i^n(q),\ i = 1, ..., d$, *and assume, for all* $n \geq 1$,

$$\sum_h \binom{n}{h} \sigma_{n-h}(p, q)(\alpha_i^h(p, q)\alpha_j^0(p, q) - \alpha_i^0(p, q)\alpha_j^h(p, q))$$

$$= \sum_{k,l,m,s} \frac{n!\sigma_{n-k-m}(p, s)\sigma_{k-l}(s, q)}{l!m!(k-l)!(n-k-m)!}(\alpha_j^m(p, s)\alpha_i^l(s, q) - \alpha_i^l(p, s)\alpha_j^m(s, q)),$$

Representations of $Ph^{\infty}(k[\underline{t}])$

I No 29

Consider the matrix,

$$D_i^n := \begin{pmatrix} \alpha_i^n(1) & r_i^n(1,2) & \dots & r_i^n(1,r) \\ r_i^n(2,1) & \alpha_i^n(2) & \dots & r_i^n(2,r) \\ . & . & \dots & . \\ r_i^n(r,1) & r_i^n(r,2) & \dots & \alpha_i^n(r)) \end{pmatrix}$$

with the relations,

$$\mathfrak{a} : r_i^0(p,q) = 0, \;\; r_i^n(p,q) = \sum_{l=0}^{n} \binom{n}{l} \alpha_i^l(p,q)\sigma_{n-l}(p,q),$$

Then $\rho(d^n t_i) = D_i^n$ define a representation,

$$\rho : Ph^{\infty}(k[\underline{t}])) \to M_r(k)$$

Proof

I No 30

Let us, as above, consider the matrix,

$$X_i = \rho(exp(\tau\delta))(t_i) = \sum_{n\geq 0} \tau^n/n! D_i^n$$

Putting

$$\alpha_i(p) = \sum_{n=0}^{\infty} \tau^n/n!\ \alpha_i^n(p),\ \alpha_i(p,q) = \sum_{n=0}^{\infty} \tau^n/n!\ \alpha_i^n(p,q),$$

and, $\sigma(p,q) = \sum_{n=0}^{\infty} \tau^n/n!\ \sigma^n(p,q)$, we find the explicite formulas,

$$X_i = \begin{pmatrix} \alpha_i(1) & \sigma(1,2)\alpha_i(1,2) & ... & \sigma(1,r)\alpha_i(1,r) \\ \sigma(2,1)\alpha_i(2,1) & \alpha_i(2) & ... & \sigma(2,r)\alpha_i(2,r) \\ ... & ... & ... & ... \\ \sigma(r,1)\alpha_i(r,1) & \sigma(r,2)\alpha_i(r,2) & ... & \alpha_i(r) \end{pmatrix},\ i = 1, ..., d.$$

Now, compute, and put, $[X_i, X_j] = 0$ and see that the condition of the theorem emerges.

Formal Moduli of Finite Representations of $Ph^{\infty}(k[\underline{t}]))$

I No 31

We may consider the *space*

$$\mathbf{A}(r) = k[\alpha_i^n(p), \sigma_n(p,q)]/\mathfrak{a},$$

with coordinates $\{\alpha_i^n(p), \sigma_n(p,q), i = 1, ..., d, n \geqslant 0, p, q = 1, ..., r\}$, and where the ideal $\mathfrak{a}$ is generated by the equations above, as the versal base space for the versal family of the non-commutative deformation theory applied to the family of $Ph^{\infty}(k[\underline{t}]))$ modules defined by the object $\mathcal{P}$.
In dimensions $d = 3, r = 3$, and order 1, the condition above reads:
$\forall p, s, q = 1, 2, 3$

$$\sigma_1(p,q)(\alpha^1(p,q) \times \alpha^0(p,q)) = -\sigma_1(p,s)\sigma_1(s,q)(\alpha^0(p,s) \times \alpha^0(s,q)).$$

This says that for any two of the three points in space, the relative momentum must sit in the plane defined by the three points, the length being determined by the 3 coupling constants. Moreover, the obvious sum of all three relative momenta must be 0.

II The generic dynamical structure induced by a metric

II No 1

Let

$$C := k[t_1, ..., t_n]$$

Then,

$$Ph(C) = k < t_1, ..., t_n, dt_1, ..., dt_n > /([t_i, t_j], [dt_i, t_j] + [t_i, dt_j]).$$

A non-degenerate metric, $g = 1/2 \sum_{i=1}^{d} g_{i,j} dt_i dt_j \in Ph(C)$ induces an isomorphism of C-modules

$$\Theta_C = Hom_C(\Omega_C, C) \simeq \Omega_C.$$

Consider the bilateral ideal (σ_g) of $Ph(C)$ generated by

$$(\sigma_g) = ([dt_i, t_j] - g^{i,j}),$$

and put,

$$C(\sigma_g) := Ph(C)/(\sigma_g).$$

The Dirac Derivation in $C(\sigma_g)$

II No 2

Let, moreover,

$$T := \sum_j T_j = -1/2 \sum_{i,j,l} \delta_{t_l}(g_{i,j}) g^{l,i} dt_j \tag{1}$$

$$= -1/2 (\sum_{k,l} \Gamma^k_{k,l} dt_l + \sum_{k,p,q} g^{k,q} \Gamma^p_{k,q} g_{p,l} dt_l), \tag{2}$$

and consider the inner derivation of $C(\sigma_g)$, defined by,

$$\delta := ad(g - T).$$

After a dull computation, we obtain, in $C(\sigma_g)$,

$$\delta(t_i) = dt_i, \ i = 1, ..., d.$$

Therefore, by universality, we have a well-defined dynamical structure, i.e. a δ stable ideal $(\sigma_g) \subset Ph^\infty(C)$, with Dirac derivation, $\delta = ad(g - T)$. It is also easy to see that (σ_g) is invariant w.r.t. isometries.

Force Laws in $C(\sigma_g)$

II No 3

There are in $Ph(C)$ force laws, one is given by,

$$
\begin{aligned}
(*)\ d^2 t_i = & -\sum_{p,q} \Gamma^i_{p,q} dt_p dt_q - 1/2 \sum_{p,q} g_{p,q}(F_{i,p} dt_q + dt_p F_{i,q}) \\
& + 1/2 \sum_{l,p,q} g_{p,q}[dt_p, (\Gamma^{i,q}_l - \Gamma^{q,i}_l)] dt_l + [dt_i, T] \\
= & -\sum_{p,q} \Gamma^i_{p,q} dt_p dt_q - \sum_{p,q} g_{p,q} F_{i,p} dt_q + 1/2 \sum_{l,p,q} g_{p,q}[F_{i,q}, dt_p] \\
& + 1/2 \sum_{l,p,q} g_{p,q}[dt_p, (\Gamma^{i,q}_l - \Gamma^{q,i}_l)] dt_l + [dt_i, T]
\end{aligned}
$$

and consistent with the Dirac derivation in $C(\sigma_g)$, as well as with the classical equation for geodesics in $C(\sigma_0)$, where we have $\sigma_0 =< \{[dt_i, t_j], [dt_i, dt_j]]\} >$, so that $C(\sigma_0) = Ph(C)_{com}$, the commutative classical phase space.

Consequences of the Force Laws, General Relativity

II No 4

We observe that we have got:
A Field Theory for connections on C-representations V, i.e. for representations,

$$\rho : C(\sigma_g) \to End_k(V)$$

with Dirac derivation for representations, $[\delta] = 0$ and Hamiltonian $Q = \rho(g - T)$.
We have also a model for *General Relativity*, for the scheme $Spec(Ph(C)_{com})$, i.e. for representations,

$$\rho : C(\sigma_0) \to End_{C(\sigma_0)}(C(\sigma_0)) \to k$$

with Dirac derivation,

$$\delta = \sum_{i,p,q} (-\Gamma^i_{p,q} v_p v_p \delta_{v_i} + v_i \delta_{t_i})$$

Here v_i are the "vertical" coordinates in the phase space of $C = k[t_1, .., t_n]$, i.e. the momenta. The Hamiltonian Q is, obviously,

Further consequences of the Force Laws

II No 5

For the Euclidean or the Minkowski metric, the Force Law, or the equation of motion, reduces to,

$$d^2 t_i = -\sum_{p,q} g_{p,q} F_{i,p} dt_q + 1/2 \sum_{l,p,q} g_{p,q} [F_{i,q}, dt_p].$$

Given a representation,

$$\rho : Ph(C) \to End_k(\Theta_C),$$

this implies the following simple evolution equation,

$$\dot{\rho}(dt_i) = \rho(d^2 t_i) = -\sum_p (F_{i,p} \rho(d_p) + 1/2 \nabla_{\delta_s}(F_{i,p})).$$

which, when we have introduced our favourite Toy Model, will give us the Bloch and the Maxwell equations.

The Space of Potentials

II No 6

Choose a particular momentum, $\rho*$, and put $\xi_i := \sum_{l=1}^{n} g^{il}\delta_l \in Der_k(C), \rho * (dt_i) =: \nabla_{\xi_i}$. Then the space of representations, ρ of $C(\sigma)$ is given as above, by

$$\rho_\psi(t_i) = t_i,\ \rho_\psi(dt_i) = \nabla_{\xi_i} + \psi_i,$$

where $\psi_i \in End_C(V)$.(On a free C-module V, one usually write ξ_i for ∇_{ξ_i}.)

The set of iso-classes is identified with the space of equivalence classes of the corresponding set of *potentials*, $\psi := (\psi_1, \psi_2, ..., \psi_n)$, naturally isomorphic to,

$$\mathcal{P} = End_C(V)^n.$$

It does not form an algebraic variety, but it has a nice structure.

The Tangent Space of $\mathcal{P}$,

II No 7

The *tangent space* $\mathbf{T}_\rho$, of $\mathcal{P}$ at $\rho_\psi : C(\sigma) \to End_C(V)$, represented by $\psi \in \mathcal{P}$, may also be identified with a quotient of $\mathcal{P}$. In fact,

$$\mathrm{Ext}^1_{C(\sigma_g)}(\rho_\psi, \rho_\psi) = Der_k(C(\sigma_g), End_k(V))/Triv.$$

Any derivation $\eta \in Der_k(C(\sigma_g), End_k(V))$, maps the relations of $C(\sigma_g)$ to zero, so we shall have,

$$[\eta(dt_i), t_j] + \rho(dt_i)\eta(t_j) - \eta(t_j)\rho(dt_i) = \eta(g^{i,j}).$$

If V is a projective C-module, such that $\mathrm{Ext}^1_C(V, V) = 0$, there exists a linear map, $\Phi_0 \in End_k(V)$, such that $\eta(t_j) = t_j\Phi_0 - \Phi_0 t_j$, for all j. We may therefore, assume all $\eta(t_i) = 0$, and then the derivation η is determined by the family of elements, $\eta(dt_i) \in End_C(V)), i = 1, ..., n$.

The Classical Gauge Invariance

II No 8

For $\rho := \rho_\psi$, corresponding to $\psi \in \mathcal{P}$, we have seen that,

$$\mathbf{T}_\rho = \mathrm{Ext}^1_{C(\sigma)}(\rho, \rho) = \mathcal{P}/\mathit{Triv}$$

where the the elements in *Triv*, corresponding to trivial derivations also mapping t_i to 0, are exactly those given by the n-tuples,

$$((\xi_1(\Phi) + [\psi_1, \Phi]), ..., (\xi_n(\Phi) + [\psi_n, \Phi])),$$

for some $\Phi \in \mathit{End}_C(V)$.

The Gauge Group $End_C(V)$

II No 9

The expression,

$$\Phi(\psi) := (\xi_1(\Phi) + [\psi_1, \Phi], ..., \xi_n(\Phi) + [\psi_n, \Phi]),$$

therefore corresponds to an infinitesimal gauge transformation,

$$\Phi \in Der_k(\mathcal{P}) =: \mathfrak{h}$$

of the space, $\mathcal{P}$, of representations of $C(\sigma_g)$, acting linearly like

$$\Phi(\rho_\psi) = (\rho_\psi) + \Phi(\psi)$$

In particular, the tangent space,

$$\mathbf{T}_{\rho_\psi} = End_C(V)^n/\mathfrak{h}.$$

The *physical* relevant space is therefore the quotient ,

$$\mathbf{P} = \mathcal{P}/\mathfrak{h}$$

of $\mathcal{P}$ with respect to the action of the Lie algebra $\mathfrak{h}$.

The Action of the Dirac Derivation on **P**

II No 10

As in the finite dimensional situation, the Dirac derivation induces a vector field

$$[\delta] \in \Theta_{\mathbf{P}},$$

as long as we, by vector field, understand any map which to an element ψ in **P** associate an element in its tangent space, i.e. in $Ext^1_{C(\sigma_g)}(V_\rho, V_\rho)$, for $\rho = \rho_0 = \psi$. It must, however, vanish at ρ, since the Dirac derivation $\delta = ad(g - T)$, necessarily must be mapped to a trivial derivation in $Der_k(C(\sigma), End_k(V))$, therefore to 0 in $Ext^1_{C(\sigma)}(V_\rho, V_\rho)$. But then it corresponds to an infinitesimal transformation of V,

$$\rho_\psi(g - T) = 1/2 \sum g_{i,j} \rho_\psi(dt_i)\rho_\psi(dt_j) + 1/2 \sum_l (\bar{\Gamma}^l_{i,l} + \Gamma^l_{i,l})\rho_\psi(dt_i)$$

This may be interpreted as saying that time acts within each representation, $\rho : C(\sigma_g) \to End_k(V)$.

Classical Notations

II No 11

The physicists usually write $\delta\psi := \Phi(\psi)$, not caring to mention Φ, taking for granted that $\delta\psi := \delta_\Phi(\psi)$ stands for an infinitesimal movement of ψ in the direction of Φ, and call the transformation above, an *infinitesimal gauge transformation*. The literature on gauge theory, and its relation to non-commutativity of space, and to quantisation of gravity, is huge. I think that the introduction of the *non-commutative phase space*, and in the metric case, the *generic* dynamical system,

$$(\sigma_g) = ([dt_i, t_j] - g^{i,j}),$$

can, to some degree, elucidate the philosophy behind this effort. See e.g. the papers, [**?**], and [**?**], where the authors initially introduce non-commutativity in the ring of *observables* generated by *coordinates*, $\hat{x}^\nu$, by imposing,

$$[\hat{x}^\nu, \hat{x}^\mu] = \Theta^{\nu,\mu},$$

where $\Theta^{i,j}$ are constants.

A Remark

II No 12

The above treatment of the notion of gauge groups and gauge transformations may also explain why, in physics, one considers *potentials* as *interaction carriers*, thus as particles *mediating force* upon other particles. And maybe one can also see why the notion of *Ghost Fields or Particles* of Faddeev and Popov, comes in. It seems to me that the introduction of ghost particles is linked to working with a particular section of the quotient map, $\mathcal{P} \to \mathbf{P}$.

The Dirac derivation, which is entirely dependent upon the notion of non-commutative phase space, is not (explicitely) found in present day physics. The parsimony principle is therefore, normally, introduced via the construction of a Lagrangian, and an Action Principle, i.e. a function of the (assumed physically significant) variables, the fields and their derivatives, defined in $\mathcal{P}$, assumed to to be invariant under the gauge transformations, so really defined in $\mathbf{P}$, and supposed to stay stable during *time* development, see [**?**].

Chern Characters and Chern-Simons Classes

II No 13

A non trivial element in the toolbox of the physicists, helping them to guess the Lagrangian, is the Chern-Simons functional, that we now spell out in some generality. Since everything we have done above is functorial, we may work on nonsingular schemes, instead of commutative k-algebras. Let us, as above, assume given a a metric g on some scheme X, and that we have an affine covering, given in terms of a family of commutative k-algebras C_α and a bundle V, defined on X, corresponding to a set of representations, $\rho_0 : C = C_\alpha \to End_k(V_\alpha)$. Let $\rho : O_X(\sigma_g) \to End_k(V)$ be a momentum at ρ_0, i.e. an extension of ρ_0 to $O_X(\sigma_g)$. Then we know that ρ induces a connection on V, and we have denoted by $\mathcal{P} := End_{O_X}(V)^d$, the set of such connections, or if we want to, the set of representations $\rho : O_X(\sigma_g) \to End_k(V)$. Recall that we may pick a *0-object* $\rho*$ in this set, such that all other representations ρ, is $\rho*$ plus a potential $\psi \in End_{O_X}(V)^d$.

Chern Characters and Chern-Simons Classes

II No 14

Consider now for any O_X-bundle ρ, V the class in $HH^n(O_X, End_k(V))$ defined by the Hochschild co-chain, $ch^n(\rho)$the k-linear map, defined locally by,

$$ch^n : C_\alpha^{\otimes n} \to End_k(V_\alpha),$$

where,

$$ch^n(c_1 \otimes c_2 ..., \otimes c_n) = \rho(dc_1 dc_2 ... dc_n) \in End_k(V_\alpha).$$

We know that ch^n is a Hochschild co-cycle, since,

$$\delta(ch^n)((c_1 \otimes c_2 ... \otimes c_{n+1})) = c_1 \rho_1((dc_2 dc_3 ... dc_{n+1})) \tag{3}$$

$$+ \sum_1^n (-1)\rho_1((dc_1 ... d(c_i c_{i+1}) ... dc_{n+1})) + (-1)^{n+1} \rho_1(dc_1 ... dc_n) c_{n+1} = 0 \tag{4}$$

Invariance of the Chern-Simons Classes

II No 15

The *Generalized Chern-Simons Class* of ρ_ψ, is then the class in the obvious double complex, defined by the covering $\{C_\alpha\}$, and the classes,

$$ch^n(\rho) \in HH^n(C_\alpha, End_k(V_\alpha)),$$

or, usually, by $1/n!\ ch^n$. Let $\Phi \in \mathfrak{h} := End_C(V)$, and consider Φ as a Hochschild 0-cocycle,

$$\Phi : k = C^0 \to End_k(V), \Phi(\alpha) = \alpha\Phi,$$

then,

$$\delta\Phi(t_i) = \rho_\psi(dt_i)\Phi - \Phi\rho_\psi(dt_i) = [\nabla_{\xi_i} + \psi_i, \Phi] = \xi_i(\Phi) + [\psi_i, \Phi].$$

This proves that

$$ch^1(\rho_\psi) = ch^1(\rho_\psi + \Phi(\psi))$$

and implies that the *Ghost Fields* have well defined Chern-Simons classes.

The Moduli Space of Metrics

II No 16

The relationship between the two different notions of time, the metric of our moduli space C of our models, and the Dirac derivative of $Ph^{\infty}(C)$ is tight.

Let $\mathbf{M}$ be the space (of isomorphism classes) of metrics on C. For every point $g \in \mathbf{M}$ consider the diagram,

$$\begin{tikzcd}
 & Ph(C) \arrow[r] \arrow[d] \arrow[dr, "\rho_1"] & Ph^2(C) \arrow[r] \arrow[d, "\rho_2"] & Ph^3(C) \ldots \arrow[r] \arrow[dl, "\rho_3"'] & Ph^{\infty}(C) \circlearrowleft \delta \\
C \arrow[ur, "d"] \arrow[r, "i"'] & C(\sigma_g) \arrow[r, "\rho"'] & End_k(V) & &
\end{tikzcd}$$

Here ρ_1 is the representation of $Ph(C)$ induced by a representation ρ of $C(\sigma_g)$.

The Moduli Space of Metrics

II No 17

Consider the family of k-algebras,

$$\mu : \mathbf{C}(\sigma) \to \mathbf{M}$$

indexed by the possible metrics of C, such that $g \in \mathbf{M}$ corresponds to $C(\sigma_g)$.
Let,

$$\mathbf{T}_{\mathbf{M},g} = \{(h_{i,j})\},\ h_{i,j} = h_{j,i} \in C.$$

be the tangent space to $\mathbf{M}$, at g. Define, $h^{i,j}$ by,

$$h_{i,j} = -\sum_{p,q} g_{i,p} h^{p,q} g_{q,j},\ h = \{h_{i,j}\} \in \mathbf{T}_{\mathbf{M}}.$$

Consider now the first order deformation $g + \epsilon h$, of the metric g, and the corresponding Dirac derivation, $ad(g + \epsilon h - T')$ in $C(\sigma_{(g+\epsilon h)})$. Then we find, in $C(\sigma_{(g+\epsilon h)})$,

$$[d' t_i, t_j] = g^{i,j} - h^{i,j}\epsilon = (g + \epsilon h)^{i,j}.$$

Metrics

II No 18

Moreover, the derivation $\eta : Ph(C) \to End_k(V)$ defined by,

$$\eta(t_i) = 0,\ \eta(dt_i) = \sum_{l,q} h^{i,l}\rho_1(g_{l,q}dt_q) = \sum_{l} h^{i,l}\nabla_{\delta_l}$$

corresponding to the 2.-order derivative of ρ, i.e. to the morphism,

$$\eta(\rho) := \rho_2 : Ph^2(C) \to End_k(V),$$

for which, $\rho_2(d^2 t_i) = \eta(dt_i)$, induces an element,

$$\eta(h) \in Ext^1_{Ph(C)}(V, V)$$

producing an injective map,

$$\eta : \mathbf{T}_{\mathbf{M},g} \to Ext^1_{Ph(C)}(V, V).$$

Gravitational Waves

II No 19

Thus, any non-trivial deformation of the metric g induces a non-trivial deformation of the $Ph(C)$-representation (ρ_1, V), and any *first order* non-trivial deformation of the $Ph(C)$-representation (ρ_1, V) induces a non-trivial deformation of the metric. Given any non-zero tangent $h \in \mathbf{T}_{\mathbf{M}_g}$, we see that there corresponds a non-zero second-order tangent of the representation ρ_0, given by the element $\eta(h) \in Ext^1_{Ph(C)}(V, V)$, determined by the ρ_1-derivation η.
This $\eta(h)$ is the measure of an acceleration of the representation ρ_0, and should correspond to a "cataclysmic change" of any massy "particle $\rho_0 : C \to End(V)$", given by $\ddot{\rho}_0 = h$, the solution of which would be a "wave", $\tilde{\rho}_0$. Combined with the structure of our "Toy Model", see [**?**], this fits well with the present understanding of gravitational waves. It also fit reasonably well with the present cosmological theory.

Gravitational Waves

II No 20

Moreover, let us, for every metric $g \in \mathbf{M}$ consider the scalar curvature R as a function on the space $\underline{C} := Spec(C)$, and fix a "compact" subset $\Omega \subset \underline{C}$. Since $R \sim 1/r^d$, where r is the "radius of curvature" of Ω at the corresponding point, it is not unreasonable to consider

$$S(g) := \int_\Omega R dv_g,$$

where dv_g is the volume element in $\underline{C}$ defined by the metric g, as the "furniture", i.e. the "material" content of the the part of space, Ω, represented by all representations of $C(\sigma_g)$. But then one could look at S, as a functional,

$$S : \mathbf{M} \to \mathbf{R}$$

the stability of which would give us a unique vector field on $\mathbf{M}$,

$$\mathfrak{G} \in \Theta_{\mathbf{M}},\ \mathfrak{G}(g) = \{\mathfrak{G}_{i,j}\} := \{R_{i,j} - 1/2 R g_{i,j}\} \in T_{\mathbf{M},g}.$$

Gravitational Waves

II No 21

Then the injective map,

$$\eta : \mathbf{T}_{\mathbf{M},g} \to Ext^1_{Ph(C)}(V, V),$$

would be a *Field Equation*, of Einstein-Hilbert type.
An "increment" h, of the metric g (in Ω) would correspond to the element of $\eta(h) \in Ext^1_{Ph(C)}(V, V)$, of first order, for all representations ρ, corresponding to an increment of energy of the total furniture of our Universe. So, again we find a Schrødinger type equation; equating derivation with respect to time, with $\mathfrak{G}$, and seeing that ρ_2 corresponds to an Hamiltonian operator on V, we find,

$$\delta_t(\rho_1) = \rho_2.$$

This is our generalised Einstein field equation, $\eta(h)$ for all ρ, being identified with the mass, energy, stress-tensor of the general relativity theory.

III Quotients in Geometry

No 1

Let $\mathfrak{g} \subset Der_k(A)$, be a sub Lie-module, and consider first, in the commutative case, the scheme (algebraic variety), $X = Spec(A)$, and the Lie algebra $\mathfrak{g}$ as a Lie algebra of vector fields defined on X. The set of maximal integral subschemes,

$$X/\mathfrak{g} := \{M \subset X | \Theta_M = \mathfrak{g}|M\},$$

is called the quotient of X by $\mathfrak{g}$, and coincides, in good cases with the quotient of X by the group of automorphisms $\mathbf{G}$ acting on X, with $\mathfrak{lie}\mathbf{G} = \mathfrak{g}$, when this exists. In the classical, commutative case, one would identify,

$$X/\mathfrak{g} = Spec(A^{\mathfrak{g}}),$$

where, $A^{\mathfrak{g}} := \{a \in A | \forall \gamma \in \mathfrak{g},\ \gamma(a) = 0\}$. This is, however, only reasonable when $\mathfrak{g}$ is reductive and/or all orbits of $\mathbf{G}$ are closed, which they rarely are.

Quotients in Non-commutative Algebraic Geometry

III No 2

In the non-commutative situation, this last definition of a quotient, has no meaning. The algebra A may have no 1-dimensional representations at all. The solution is to define the relevant points $Simp(A)$, of the geometry $\mathbf{A}$, defined by A, and then to seek out those points that should correspond to the points of the quotient $\mathbf{A}/\mathfrak{g}$.

Consider a representation $\rho : A \to End_k(V)$, and let for every $\gamma \in \mathfrak{g}$ the derivation, $\gamma \circ \rho \in Der_k(A, Hom_k(V, V))$, map to 0 in $Ext^1_A(V, V)$. This means that the representation ρ, is not moved by the action of $\mathfrak{g}$. As above, it follows from, $\kappa(\gamma \circ \rho) = 0$, that there exist an element, $Q_\gamma \in Hom_k(V, V)$, the Hamiltonian, such that for all $\gamma \in \mathfrak{g}$ and all $a \in A$,

$$\rho(\delta(a)) = Q_\gamma \circ \tilde{\rho}(a) - \rho(a) \circ Q_\gamma.$$

which can be written as,

$$Q_\gamma(av) = \gamma(a)v + aQ_\gamma(v), \forall v \in V,$$

i.e. Q is a $\mathfrak{g}$-connection on V.

Quotients in Non-commutative Algebraic Geometry

III No 3

Now, we define,

$$\mathbf{A}/\mathfrak{g} := Simp(\{\rho|\delta := \gamma \circ \rho = 0 \in Ext^1_A(V, V)\})$$

where *Simp* of course picks out those representations of this sort, with no such sub-representations.
The aim of, my kind, of non-commutative algebraic geometry is to to associate to any family of representations $\mathbf{V}$, of the algebra A, the optimal extension-algebra,

$$\eta : A \to \mathbf{O}(\mathbf{V})$$

for which the family $\mathbf{V}$ becomes the set of simple $\mathbf{O}(\mathbf{V})$-representations.
Then $\mathbf{V}$ should be called: A Scheme for $\mathbf{O}(\mathbf{V})$.

Quotients in Geometry and Physics: Gauge Groups

III No 4

The notion of Gauge Group in physics, is intimately related to this non-commutative quotient structure.
We shall, in fact, show that there is an algebra H, representing our Cosmos, a Lie algebra, $\mathfrak{G}$ acting on $H(\sigma) := Ph(H)/([h, dh]|h \in H)$, the partially commutativization of $Ph(H)$, such that the main ingredients of the Standard Model, including representations like the Weyl and the Dirac Spinors pop up as points of the quotient,

$$H(\sigma)/\mathfrak{G},$$

suggesting that the standard Quantum Theory should be thought of as part of non-commutative algebraic geometry.

Global Gauge Groups

III No 5

Suppose, we have identified a k-Lie algebra $\mathfrak{g}_0 \subset Der_k(A)$, of infinitesimal automorphisms, i.e. of derivations of A, a *global gauge groupe*, leaving invariant the physical properties of our phenomena $\mathbf{P}$. We would then be led to consider the *quotient space* $\mathbf{M}/\mathfrak{g}_0$, which in our non-commutative geometry, is equivalent to restricting our representations, $\rho : A \to End_k(V)$, to those representations V for which, $\mathfrak{g}_0 \subset \mathfrak{g}_V$. This would then imply that the corresponding *Hamiltonians*, Q_γ define a $\mathfrak{g}_0$-connection on V,

$$Q : \mathfrak{g}_0 \longrightarrow End_k(V),$$

such that, for all $c \in A$, and for all $\gamma \in \mathfrak{g}_0$, $\rho(\gamma(c)) = [Q_\gamma, \rho(c)]$.

Global Gauge Groups and Forces

III No 6

This is usually written,

$$\rho(\gamma(c)) = [\gamma, \rho(c)].$$

The curvature,

$$R(\gamma_1, \gamma_2) := [Q_{\gamma_1}, Q_{\gamma_2}] - Q_{[\gamma_1,\gamma_2]} \in End_A(V),$$

corresponds to a *global force* acting on the representation ρ. These forces, *mediated* by the *gauge-particles*, $\lambda \in \mathfrak{g}_0$, will be the first to be studied in some details. Put,

$$Rep(A, \mathfrak{g}_0) := \{\rho \in Rep(A)|,\ \kappa(\gamma\rho) = 0, \forall \gamma \in \mathfrak{g}_0\} = \{\rho \in Rep(A)|\mathfrak{g}_0 \subset \mathfrak{g}_\rho\},$$

where $Rep(A)$ is the category of all representations of C, and notice that, in the commutative situation, if we consider the case where the gauge group $\mathfrak{g}_0 = Der_k(A)$ then $Rep(A, \mathfrak{g}_0)$ is the category of *A-Connections*, for which the space of isomorphism classes is discrete. Notice that this is the situation in the classical quantum theory, where the *Hilbert Space* is always considered as the unique state space of interest.

Simple Objects

III No 7

An object $V \in Rep(A, \mathfrak{g}_0)$ is called simple if there are no non-trivial sub-objects of V in $Rep(A, \mathfrak{g}_0)$. The generalized quotient $Simp(A)/\mathfrak{g}_0$, is by definition, the set, $Simp(A : \mathfrak{g}_0)$, of iso-classes of simple objects in $Rep(A, \mathfrak{g}_0)$.

If the curvature also vanish, there is a canonical homomorphism,

$$\phi : U(\mathfrak{g}_0) \to End_k(V).$$

where $U(\mathfrak{g}_0)$ is the universal algebra of the Lie algebra $\mathfrak{g}_0$.

Gauge Representations

III No 8

In the general case let,

$$A'(\mathfrak{g}_0) \subset End_k(A),$$

be the sub-algebra generated by A and $\mathfrak{g}_0$. Then we put,

$$A(\mathfrak{g}_0) = A'(\mathfrak{g}_0)/(\gamma a - a\gamma - \gamma(a))$$

and we have an identification between the set of $\mathfrak{g}_0$-connections on V, and the set of k-algebra homomorphisms,

$$\rho_{\mathfrak{g}} : A(\mathfrak{g}_0) \to End_k(V),$$

since any such would respect the relation above, such that,

$$\rho_{\mathfrak{g}_0}(\gamma a) = \rho_{\mathfrak{g}_0}(\gamma)\rho_{\mathfrak{g}}(a) = \rho_{\mathfrak{g}_0}(a)\rho_{\mathfrak{g}_0}(\gamma) + \rho_{\mathfrak{g}_0}(\gamma(a)).$$

Gauge Representations and Quotients

III No 9

In the above situation, we have the following isomorphisms,

$$Simp(A)/\mathfrak{g}_0 := Simp(A : \mathfrak{g}_0) \simeq Simp(A(\mathfrak{g}_0)).$$

Notice that the commutant in $A(\mathfrak{g}_0)$, of $\mathfrak{g}_0$ is the subring,

$$A^{\mathfrak{g}_0} := \{a \in A | \forall \gamma \in \mathfrak{g}_0, \gamma(a) = 0\} \subset A.$$

Notice also that the commutativisation, $A(\mathfrak{g}_0)^{com}$, of $A(\mathfrak{g}_0)$ is the quotient of $A(\mathfrak{g}_0)$ by an ideal containing $\{\gamma(a) | a \in A, \gamma \in \mathfrak{g}_0\}$. Therefore there is a natural map,

$$A^{\mathfrak{g}_0} \to A(\mathfrak{g}_0)^{com}.$$

However this map may not be injective, so we cannot, in general, identify the rank 1 points of $Simp(A, \mathfrak{g}_0)$, with $Simp_1(A^{\mathfrak{g}_0})$.

The Commutative Classical Case

III No 10

If $A = C$ is assumed commutative, the classical invariant theory identifies the two schemes, $Spec(C)/\mathfrak{g}_0$ and $Spec(C^{\mathfrak{g}_0})$, which in the above light, is not entirely kosher. However, if $\mathfrak{a} \subset C$ is an ideal, stable under the action of $\mathfrak{g}_0$, then since any derivation γ of C acts on the multiplicative operators $a \in C$ as $\gamma(a) = \gamma a - a\gamma$, it is clear that the quotient $C/\mathfrak{a}$ is an $(C, \mathfrak{g}_0)$-representation. Moreover,

$$C/\mathfrak{a} \in Simp_1(C)/\mathfrak{g}_0$$

if and only if the subset $Simp_1(C/\mathfrak{a}) \subset Simp_1(C)$ is the closure of a *maximal integral subvariety* for $\mathfrak{g}_0$. The *space* of such integral subvarieties is what we, previously have termed the *non-commutative quotient*, $Spec(C)/\mathfrak{g}_0$.

IV Local Gauge Groups

No 1

Suppose now there is a C-Lie algebra $\mathfrak{g}_1$, acting C-linearly on those C-modules V, which we would consider of *physical* interest. $\mathfrak{g}_1$ would then be called a local gauge group. One may then want to know wether the given action of $\mathfrak{g}_0$ moves $\mathfrak{g}_1$ in its formal moduli as C-Lie algebra. If so, the action of $\mathfrak{g}_1$ would not be invariant under the the gauge transformations induced by $\mathfrak{g}_0$, and we should not consider $(\rho, \mathfrak{g}_1)$ as physically kosher. If, on the other hand, the action of $\mathfrak{g}_0$ does not move $\mathfrak{g}_1$ in its formal moduli, it should follow that there is a relation between the $\mathfrak{g}_0$-action (i.e. the connection) on V, and the action of $\mathfrak{g}_1$. Now, in the case $\mathfrak{g}_0 \subset Der_k(C)$, it follows from the Kodaira-Spencer map,

$$ks : Der_k(C) \to A^1(C, \mathfrak{g}_1 : \mathfrak{g}_1),$$

see Lemma(2.3), of [6], that the following holds,

Quotients of Local Gauge Groups

IV No 2

Assume $\mathfrak{g}_1$ as a C-module is such that $\mathfrak{g}_0 \subset \mathfrak{g}_{\mathfrak{g}_1}$, and let $c_{i,j}^k \in C$ be the structural constants of $\mathfrak{g}_1$ with respect to some C-basis $\{x_i\}$, and let $\delta : \mathfrak{F} \to \mathfrak{g}_1$ be a surjective morphism of a free C-Lie algebra $\mathfrak{F}$ onto $\mathfrak{g}_1$, mapping the generators $\mathfrak{x}_i$ of $\mathfrak{F}$ onto x_i. Let $\mathfrak{F}_{i,j} = [\mathfrak{x}_i, \mathfrak{x}_j] - \sum_k \mathfrak{c}_{i,j}^k \mathfrak{x}_k \in ker\delta$, and let $\gamma \in Der_k(C)$. Then, $ks(\gamma)$ is the element of $A^1(C, \mathfrak{g}_1 : \mathfrak{g}_1)$ determined by the element of $Hom_{\mathfrak{F}}(ker(\delta), \mathfrak{g}_1)$, given by the map,

$$\mathfrak{F}_{i,j} \to -\sum_k \gamma(c_{i,j}^k) x_k.$$

For $ks(\gamma)$ to be 0, there must exist a C- derivation $D_\gamma : \mathfrak{F} \to \mathfrak{g}_1$, (a potential), such that

$$D_\gamma(\mathfrak{F}_{i,j}) = D_\gamma([\mathfrak{x}_i, \mathfrak{x}_j] - \sum_k \mathfrak{c}_{i,j}^k \mathfrak{x}_k) = -\sum_k \gamma(c_{i,j}^k) x_k$$

Quotients of Local Gauge Groups

IV No 3

Let now $\nabla : \mathfrak{g}_0 \to Der_k(\mathfrak{g}_1)$ be a connection on the C-module $\mathfrak{g}_1$. Then,

$$ks(\gamma) = 0$$

iff there exist a $\mathfrak{g}_0$-Lie-connection of the form $\mathfrak{D} = \nabla - D$ on $\mathfrak{g}_1$, i.e. a k-linear map,

$$\mathfrak{D} : \mathfrak{g}_0 \to Der_k(\mathfrak{g}_1),$$

such that, for $\gamma \in \mathfrak{g}_0,\ c \in C,\ \kappa \in \mathfrak{g}_1$,

$$\mathfrak{D}_\gamma(c \cdot \kappa) = c\mathfrak{D}_\gamma(\kappa) + \gamma(c) \cdot \kappa.$$

Lie-Cartan Pairs

IV No 4

If the curvature,

$$R(\gamma_1, \gamma_2) := [\mathfrak{D}\gamma_1, \mathfrak{D}\gamma_2] - \mathfrak{D}[\gamma_1, \gamma_2]$$

representing the 2. order action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$, (the force, as a physicist might have said), vanish, the map,

$$\mathfrak{D} : \mathfrak{g}_0 \to Der_k(\mathfrak{g}_1),$$

would be a Lie-algebra morphism. And then this notion is related to the notion of a Lie-Cartan pair. In fact, the structure given by $\mathfrak{D}$ defines a Lie algebra structure on the sum

$$\mathfrak{g} := \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

with Lie-products of the sum defined as the product in each Lie algebra, and the cross-products defined for, $\gamma \in \mathfrak{g}_0, \xi \in \mathfrak{g}_1$, as,

$$[\gamma, \xi] = \mathfrak{D}_\gamma(\xi),$$

Lie Algebroids

IV No 5

A structure like this, a Lie-Cartan pair, is now often called a Lie algebroid, and the operation of the C-Lie algebra $\mathfrak{g}_1$ is, when it is supposed to be physically interesting, called a *local* gauge group.
The category of representations $\rho : C \to End_k(V)$ with this property vis a vis the Lie algebra $\mathfrak{g}$, should be called $\mathfrak{g}$-*insensitive*, and the space of simple $\mathfrak{g}$-insensitives shall be denoted.

$$Simp(A)/\mathfrak{g}.$$

This is going to be our central object of study.

V The Role of Cartan sub-Algebras in Physics

IV No 6

Physicists have a way of classifying, or naming *states*, i.e. the elements of the representation vector space V, according to certain numbers associated to them, like spin, charge, hyperspin, etc. We find this in the situation above, as follows.

Consider the Cartan algebra $\mathfrak{h} \subset \mathfrak{g}_1$. It will operate on the above representation space V as diagonal matrices, and the eigenvectors may be labeled by the corresponding eigenvalues.

Notice also that if $V_i \in Rep(C, \mathfrak{g}), i = 1, 2$, then it follows from (1.15) that an extension of the $C(\mathfrak{g}_0)$-module V_1 with V_2 will also sit in $Rep(C, \mathfrak{g})$.

Put, as above, $A := C(\sigma)$ where σ is a dynamical structure, and assume given an action of the Lie algebra $\mathfrak{g}_0$ on C. Notice that since $Ph()$ is a functor in the category of algebras and algebra morphisms, the action of $\mathfrak{g}_0$ extends to $Ph(C)$, but not necessarily to a dynamical system of the type $C(\sigma)$.

This will turn out to be important for our version of the Standard Model.

VI Deformations of associative algebras

V No 1

We fix a field k. All algebras occurring, will be associative k-algebras. A deformation of an algebra A parametrized by the (commutative) k-algebra, B, is a flat k-algebra homomorphism, $B \to A_B$, such that there is a "point", i.e. a homomorphism, $\rho : B \to k$, such that $k \otimes_B A_B \simeq A$.
For any associative k-algebra A, there is a formal moduli, i.e. a complete local k-algebra, $\mathbf{H}(A)$, and a deformation, a versal family $\mu : \mathbf{H}(A) \to \tilde{\mathbf{A}}$, of associative algebras, such that for any other deformation $B \to A_B$, with B a finite dimensional local k-algebra, there is a diagram,

$$\begin{tikzcd} \mu : \mathbf{H}(A) \arrow[r] \arrow[d] & \tilde{\mathbf{A}} \arrow[d] \\ B \arrow[r] & A_B \end{tikzcd}$$

with $A_B \simeq B \otimes_{\mathbf{H}(A)} \tilde{\mathbf{A}}$.

The tangent space $T^\star$ of the formal moduli of an associative algebra

VI No 2

The tangent space of $\mathbf{H}(A)$, i.e. the dual k vectorspace of $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal of $\mathbf{H}(A)$, is calculated as:

$$T^\star = A^1(k, A; A) = Hom_F(ker(\rho), A)/Der$$

where, $\rho : F \to A$, is a surjective homomorphism of a free k-algebra F onto the given algebra A, Hom_F is the set of F-bilinear maps, and $Der \subset Hom_F$, denotes the restrictions to $ker(\rho)$, of the derivations $Der_k(F, A)$.

Deformations of U as associative algebras

VI No 3

- $A = k[x_1, x_2, x_3]$ is the commutative coordinate ring of the affine 3-space.
- $U = A/(\underline{x})^2$ is, geometrically, a thick point in affine 3-space, but
- U is also a quotient of the free associative k-algebra, $F = k < x_1, x_2, x_3 >$,
- Let $\rho : F \to U$ be the quotient map, then the kernel, $ker(\rho) = (x_i x_j),\ i, j = 1, 2, 3$

A deformation of U parametrized by the (commutative) k-algebra, B, is a flat k-algebra homomorphism,
$B \to B < x_1, x_2, x_3 > /(x_i x_j + \sum b^l_{i,j} x_l + b^0_{i,j})$
It is easy to compute the tangent space of the formal moduli of U,

$$dim_k A^1(k, U; U) = dim_k Hom_F(ker(\rho), U)/Der = 27.$$

The restricted formal moduli of U

VI No 4

Pick any two 3-vectors,

$$\overline{o} := (o_1, o_2, o_3), \overline{p} := (p_1, p_2, p_3).$$

The bi-linear homomorphisms,

$$\kappa : ker(rho) = (x_i x_j) \to U, \ \kappa(x_i x_j) := o_i x_j + x_i p_j$$

represents linearly independent elements in $A^1(k, U; U)$, and span a subspace of the tangent space of the formal moduli of U generating a quotient of the versal family,

$$\begin{tikzcd} \mu : \mathbf{H}(U) \arrow[r] \arrow[d] & \tilde{\mathbf{U}} \arrow[d] \\ H(U) \arrow[r] & \mathbf{U} \end{tikzcd}$$

The restricted versal family

VI No 5

- Let $\underline{o} := (o_1, o_2, o_3)$, and $\underline{p} := (p_1, p_2, p_3)$ be sets of independent coordinates, and let,
- $H(U) =: H = k[o_1, o_2, o_3, p_1, p_2, p_3]$
- Put, $\underline{H} := Spec(H) = \mathbf{A}^3 \times \mathbf{A}^3$, and we have got a deformation of U,

$$\tilde{\rho} : H \to \mathbf{U} := H < x_1, x_2, x_3 > /(x_i x_j - o_i x_j - x_i p_j + o_i p_j)$$

  parametrized by the 6-dimensional scheme $\underline{H}$.
- Let $o := (o_1, o_2, o_3)$, and $p := (p_1, p_2, p_3)$ be two vectors, then $U(o, p) := k < x_1, x_2, x_3 > /(x_i x_j - o_i x_j - x_i p_j + o_i p_j)$
- Put, $B = k[t], b^l_{i,j} = \epsilon_{i,j,l} t,\ b^0_{i,j} = \delta_{i,j}$ then the deformation of U along t for $t \neq 0$ is constant, equal to the Quaternions.

VIIThe Toy Model

No 1

My favourite "Toy Model", of General Relativity, and Quantum Theory is the philosophically reasonable (?) Physical Model, of an Observer and an Observed in 3-dimensional space, mathematically modelled by the Hilbert scheme **H** of length 2 sub-schemes in $\mathbf{A}^3$. Consider the diagonal, $\underline{\Delta} \subset \mathbf{A}^3 \times \mathbf{A}^3 = \underline{H}$, and let $\underline{\tilde{H}}$ be the blow up of $\underline{H}$ in $\underline{\Delta}$. We find a diagram,

$$\begin{tikzcd} \underline{\tilde{H}} \arrow[r] \arrow[d, "Z_2"] & \underline{H} \supset \underline{\Delta} & \underline{\tilde{\Delta}} \subset \underline{\tilde{H}} \arrow[l] \\ \mathbf{H} & & \end{tikzcd}$$

where $\mathbf{H} = Hilb^2(\mathbf{A}^3) = \underline{\tilde{H}}/Z_2$. Call he generator, P of Z_2, the Parity operator.

Coordinates for **H**: $(\underline{\lambda}, \underline{\omega}, \rho)$, with $\underline{\lambda} \in \underline{\Delta}$, $\underline{\omega}$ a spherical coordinate of the blow-up of $\underline{\Delta}$ in $\underline{H}$, and, ρ the length from $\underline{\Delta}$ along the line defined by $\underline{\omega}$.

Coordinates of $\underline{\tilde{H}}$

VII No 2

This is a convenient parametrisation of $\underline{\tilde{H}}$. Consider, as above, for each $\underline{t} \in \underline{\tilde{H}}$ the length ρ, in $\mathbf{E}^3$, the Euclidean space, of the vector (o, p). Given a point $\underline{\lambda} \in \underline{\Delta}$, and a point $\xi \in E(\underline{\lambda}) = \pi^{-1}(\underline{\lambda})$, of the fiber of,

$$\pi : \underline{\tilde{H}} \to \underline{H},$$

at the point $\underline{\lambda}$, for $o = p$. Since $E(\underline{\lambda})$ is isomorphic to S^2, parametrized by $\underline{\omega} = (\phi, \theta)$, any element of $\underline{\tilde{H}}$ is now uniquely determined in terms of the triple $\underline{t} = (\underline{\lambda}, \underline{\omega}, \rho)$, such that $c(\underline{t}) := c(o, p) = \underline{\lambda}$, and such that ξ is defined by the line $\underline{op}$, and the action of θ keeps ξ fixed. Here $\rho \geq 0$, and notice that, at the exceptional fiber, i.e. for $\rho = 0$, the momentum corresponding to $d\rho$ is not defined.

Metrics or Time

VII No 3

Any metric, and therefore also the notion of Time of the Model $\underline{\tilde{H}}$, can be given the form,

$$g = h_\rho(\underline{\lambda}, \underline{\phi}, \rho) d\rho^2 + h_{\underline{\phi}}(\underline{\lambda}, \underline{\phi}, \rho) d\underline{\phi}^2 + h_{\underline{\lambda}}(\underline{\lambda}, \underline{\phi}, \rho) d\underline{\lambda}^2,$$

where $d\underline{\phi}^2$ is the natural metric in $S^2 = E(\underline{\lambda})$. Pick the simplified space, in which $\underline{\omega}$ is reduced to the angle ϕ, and the coordinates $\underline{\lambda}$ is reduced to one parameter $\lambda = |\underline{\lambda}|$.

$$g = (\frac{\rho - h(\lambda)}{\rho})^2 d\rho^2 + (\rho - h(\lambda))^2 d\phi^2 + \kappa(\lambda) d\lambda^2,$$

This correspond to considering the sub-universe of $\mathbf{M}(BB)$, parametrized by (λ, ϕ, ρ).

Big Bang Model

VII No 4

Computing the geodesics (the Force Law), taking into account that Time is the metric, we find,

$$\frac{d^2\lambda}{dt^2} = -(\frac{\rho - h(\lambda)}{\rho})(\frac{1}{\kappa(\lambda)})(\frac{dh}{d\lambda})(\frac{d\rho}{dt})^2$$
$$- (\rho - h(\lambda))(\frac{1}{\kappa(\lambda)})(\frac{dh}{d\lambda})(\frac{d\phi}{dt})^2$$
$$+ (\frac{dlog(\kappa)}{d\lambda})(\frac{d\lambda}{dt})^2$$

Big Bang Model

VII No 5

Moreover we find,

$$\frac{d^2\rho}{dt^2} = -(\frac{h(\lambda)}{\rho(\rho - h(\lambda))})(\frac{d\rho}{dt})^2$$
$$+ (\frac{2}{(\rho - h(\lambda))})(\frac{dh}{d\lambda})(\frac{d\rho}{dt})(\frac{d\lambda}{dt}) + (\frac{\rho^2}{(\rho - h(\lambda))})(\frac{d\phi}{dt})^2,$$
$$\frac{d^2\phi}{dt^2} = -2/(\rho - h(\lambda))\frac{d\rho}{dt}\frac{d\phi}{dt} + 2/(\rho - h(\lambda))(\frac{dh}{d\lambda})(\frac{d\phi}{dt})(\frac{d\lambda}{dt})$$

where t, is the time parameter of the model. From these formulas we see that the Gravitation is expanding inside the Horizon and contracting outside. Conservation of mass implies, $h(\underline{\lambda}) = h_0/\lambda$. From this follows that the Horizon at the BB, i.e. for $\lambda = 0$, is all of space. Interpreting λ as Cosmological time we find a striking cosmological model, complete with Inflation and Hubble formulas, $v = r/t$ and $v/\sqrt{1 - v^2} = r/\lambda$.

Kepler

VII No 6

Now let $h(\underline{\lambda}) = h$ be constant, then the geodesics have the equations,

$$\frac{d^2\rho}{dt^2} = -(\frac{h}{\rho(\rho-h)})(\frac{d\rho}{dt})^2 + (\frac{\rho^2}{(\rho-h)})(\frac{d\phi}{dt})^2,$$
$$\frac{d^2\phi}{dt^2} = -2/(\rho-h)\frac{d\rho}{dt}\frac{d\phi}{dt}, \quad \textit{Kepler's 2.Law.}$$
$$\frac{d^2\lambda}{dt^2} = 0,$$

The definition of time gives us,

$$\rho^{-2}(\frac{d\rho}{dt})^2 = (\rho-h)^{-2}K^2 - (\frac{d\phi}{dt})^2.$$

where, $K^2 = (1 - (\frac{d\lambda}{dt})^2)$, is the kinetic energy.

Kepler and Newton

VII No 7

Put this into the first equation above, and obtain,

$$\frac{d^2\rho}{dt^2} = -hK^2(\frac{\rho}{\rho - h})\frac{1}{(\rho - h)^2} + (\frac{\rho + h}{\rho - h})\rho(\frac{d\phi}{dt})^2.$$

Assume now $r := \rho - h \approx \rho$, we find,

$$\frac{d^2 r}{dt^2} = -\frac{hK^2}{r^2} + r(\frac{d\phi}{dt})^2, \ \textit{Kepler's 1.Law.}$$

The constant h, the radius of the exceptional fibre, is thus also related to mass. Recall that the Schwarzschild radius, the Einstein equivalent to h, is assumed to be,

$$r_s = 2GM/c^2,$$

where, G = Newton's gravitational constant, M = mass, c = speed of light, which here, of course, is put equal to 1.

VIII $\mathbf{U}$ extends to $\mathbf{H}$

No 1

For every $(o,p) \in \mathbf{A}^3 \times \mathbf{A}^3 = \underline{H}$, there is an associative 4-dimensional k-algebra $U(o,p)$, the fibre of $\mathbf{U}$ at (o,p). Moreover, if $o = p$, then $U(o,p) \simeq U$. We obtain the extended diagram,

$$\begin{tikzcd}
\underline{\tilde{H}} \arrow[r] \arrow[d, "Z_2"] & \underline{H} & \underline{\Delta} \arrow[l] & \underline{\tilde{\Delta}} \subset \underline{\tilde{H}} \arrow[l] \\
\mathbf{H} & \mathbf{U} \arrow[l] \arrow[u] & U \arrow[l] \arrow[u] &
\end{tikzcd}$$

This follows from the following elementary relations: For $(o,p) \in \underline{H}$, for every $c \in \mathbf{A}^3$ and for any non-zero $\kappa \in k$, we have,

- $U(\kappa o, \kappa p) \simeq U(o,p)$
- $U(o,p) \simeq U(o-c, p-c)$
- $U(-p,-o) \simeq U(o,p)$

A Universal Gauge Group

VIII No 2

Consider the bundle of Lie algebras, defined on $\mathbf{H}$ by,

- $\mathfrak{g} := Der_{\mathbf{H}}(\mathbf{U})$
- $\mathfrak{g}(o,p) = Der_{k(o,p)}(U(o,p)), (o,p) \in \underline{H}$.

Any element $\delta \in \mathfrak{g}(o,p)$ must have the form,
$\delta(x_i) = \delta_i^0 + \delta_i^1 x_1 + \delta_i^2 x_2 + \delta_i^3 x_3$. Consider the 4-vectors,
$\delta_i = (\delta_i^0, \delta_i^1, \delta_i^2, \delta_i^3)$, $\overline{o} = (1, o_1, o_2, o_3)$, $\overline{p} = (1, p_1, p_2, p_3)$

Theorem

- $\delta \in \mathfrak{g}(o,p)$ *if and only if* $\delta_i.\overline{o} = \delta_i.\overline{p} = 0$,
- *If* $o \neq p$, *then,* $\mathfrak{g}(o,p) \simeq \begin{pmatrix} 0 & \star & \star \\ 0 & \star & \star \\ 0 & \star & \star \end{pmatrix}$
- $rad(\mathfrak{g}) = \{u, r_1, r_2\}$, $\mathfrak{g}/rad \simeq \mathfrak{sl}(2) = \{h, e, f\} \subset \mathfrak{g}$
- $h \in \mathfrak{h} \subset \mathfrak{g}$; *the generator of the Cartan algebra.*

Canonical Basis for Θ_H

VIII No 3

Notice also that, in this case, the unique 0 (resp light)-velocity tangent line at the point $\underline{t}_0 = (o, p)$, $o = (0,0,0), p = (1,0,0)$, killed by $\mathfrak{g}$, is represented by,

- $d_3 := ((1,0,0),(1,0,0))$, the unique 0-velocity, resp.
- $c_3 := ((1,0,0),(-1,0,0))$, the unique light-velocity.

Let, $d_1 := ((0,1,,0),(0,1,0))$, $d_2 := ((0,0,1),(0,0,1))$ and let, $c_1 := ((0,1,,0),(0,-1,0))$, $c_2 := ((0,0,1),(0,0,-1))$,.

- Then $\{c_1, c_2, c_3, d_1, d_2, d_3\}$ is a basis for the tangent space $\Theta_{\underline{t}_0}$, $\{c_1, c_2, c_3\}$ for $\tilde{c}_{\underline{t}_0}$, and $\{d_1, d_2, d_3\}$ for $\tilde{\Delta}_{\underline{t}_0}$.

Put, $B_o :=< c_1, d_1 >$ and $B_p :=< c_2, d_2 >$.

Spin and Isospin

VIII No 4

Denote by $\tilde{\Delta} \subset \tilde{\Theta}_{\underline{\tilde{H}}}$ the 3-dimensional distribution, generated by the translations $\{(o,p) \to (o+c, p+c),\ c \in \mathbf{A}^3\}$, and complexify all bundles. Introduce a Riemanian metric g on the space $\mathbf{H}$, and see that we have the following structures uniquely defined,

- $\tilde{\Delta}_{\mathbf{C}}$, with the action of $\mathfrak{su}(3)$
- $\Theta_{\underline{\tilde{H}}} \simeq B_o \oplus B_p \oplus A_{o,p} \simeq \tilde{\Delta} \oplus \tilde{c}$; resp. 0-and light-velocities.
- There exists a canonical action of $\mathfrak{g}_{\mathbf{C}}$ on $\Theta_{\tilde{H},\mathbf{C}}$, respecting the decomposition, such that,
- $\mathfrak{g}(o,p)$ acts on the tangent space $T_{\mathbf{H},(o,p)} = T_{\mathbf{A}^3,o} \times T_{\mathbf{A}^3,p}$ killing the vector $p - o$, in both factors.

The canonical action of these principal Lie bundles on the complexified tangent bundle $\Theta_{\mathbf{H}}$ "determins all Fields".

Action of the Gauge group

VIII No 5

The generators, $h, e, f \in \mathfrak{sl}(2) \subset \mathfrak{g}$ act, in the above basis, like,

$$h = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Action of the Gauge group

VIII No 6

$$f = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The generators, $u, r_1, r_2 \in rad(\mathfrak{g})$ act, in the above basis, like,

$$u = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Action of the Gauge group

VIII No 7

$$r_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

$$r_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Action of the Gauge group

VIII No 8

We observe here that the rotations (spin) around the 3 different axes in $\tilde{c}$, respectively in $\tilde{\Delta}$, are given by, $(e-f),(e-r_1),(f-r_2)$, about the axe defined by $(op)=c_3,c_2,c_1$, respectively $(op)=d_3,d_2,d_1$,
Recall that $\tilde{\Delta}\subset\Theta_{\tilde{H}}$, is the sub-bundle defined, at the point $\underline{t}$ as the space of tangents of the form (ξ,ξ). Given a metric on $\underline{\tilde{H}}$, we may look at the action of $\mathfrak{su}(3)$ on $\tilde{\Delta}\otimes\mathbf{C}$. Knowing that $\Theta=\tilde{c}\oplus\tilde{\Delta}$, $\mathfrak{su}(3)$ acts in the obvious way on the lower right corner, like

$$\gamma=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&*&*&*\\0&0&0&*&*&*\\0&0&0&*&*&*\end{pmatrix}.$$

Action of the Gauge group

VIII No 9

Since the 0-velocity direction defined at (o, p), is $(o - p, o - p)$, which here is d_3, we may in an essential unique way decompose the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{su}(3)$, into the Cartan subalgebra $\mathfrak{h}_1$, for the $\mathfrak{su}(2)$-component leaving δ_3 invariant, and the part $\mathfrak{h}_2 \subset \mathfrak{h}$ perpendicular, in the Killing metric, to $\mathfrak{h}_1$.

$$\mathfrak{h}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathfrak{h}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2/3 \end{pmatrix},$$

Classically one denotes the other 6 base elements of $\mathfrak{su}(3) \otimes \mathbf{C}$, as, $\mathfrak{e}^i_{\pm}, i = 1, 2, 3$.

Action of the Gauge group

VIII No 10

The restriction to $\tilde{\Delta}$ of these operators, in the basis $\{d_1, d_2, d_3\}$ are given by,

$$\mathfrak{e}_+^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathfrak{e}_+^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \mathfrak{e}_+^3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

and their duals,

$$\mathfrak{e}_-^1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathfrak{e}_-^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathfrak{e}_-^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Commutators in $\mathfrak{su}(3)$

VIII No 11

We need the commutators, $[\mathfrak{h}_1, \mathfrak{h}_2] = 0$, and,

$$[\mathfrak{h}_1, \mathfrak{e}^1_\pm] = \pm\mathfrak{e}^i_\pm, [\mathfrak{h}_1, \mathfrak{e}^2_\pm] = \mp 1/2\mathfrak{e}^2_\pm, [\mathfrak{h}_1, \mathfrak{e}^3_\pm] = \pm 1/2\mathfrak{e}^3_\pm$$

together with the following ones,

$$[\mathfrak{h}_2, \mathfrak{e}^1_\pm] = 0, [\mathfrak{h}_2, \mathfrak{e}^2_\pm] = \pm\mathfrak{e}^2_\pm, [\mathfrak{h}_2, \mathfrak{e}^3_\pm] = \pm\mathfrak{e}^3_\pm$$

Notice, for later use, that the only non-zero products of the $\mathfrak{e}^i_\pm$ are,

$$\mathfrak{e}^1_+ \cdot \mathfrak{e}^2_+ = \mathfrak{e}^3_+, \mathfrak{e}^2_- \cdot \mathfrak{e}^1_- = \mathfrak{e}^3_-.$$

IX Elementary Particles

No 1

Together these formulas show that the quotients of of the $\mathfrak{g}*$-representation $\Theta_{\tilde{H}}$ are the following,

- $\tilde{c}$ and therefore also the *photon* $\{c_1, c_2\}$ and a singleton, $\{c_3\}$, both simple.
- $\tilde{\Delta}$ and therefore the *electron* $\{d_1, d_2\}$ and a singleton, $\{d_3\}$, both simple.
- Weyl spinors, B_o, B_p, and Dirac spinors, $B_o \oplus B_p$

The non-trivial simple quotients of of the $\mathfrak{su}(3)$-representation $\Theta_{\tilde{H}}$ are reduced to,

- The quarks, $\tilde{\Delta}$

Photons

IX No 2

We observe that the action of $\phi \in u(1)$ is, $exp(\imath\phi \cdot (e - f))$, on the the (transverse) light-wave is given by,

$$A(\phi) = \begin{pmatrix} cos\phi & -sin\phi & 0 & 0 & 0 & 0 \\ sin\phi & cos\phi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is clear that this, together with the formulas above give good reasons to believe that there is a relation between this model, and the Standard Model (and so also to the 8-fold way of Gell-Mann). Moreover, here all ingredients are universally given by the information contained in the singularity U, the Big Bang, in my tapping. The choice of metric depends, however, on the nature of what I have called the Furniture of the model.

The Pauli Matrices

IX No 3

It is now easy to see that the the Pauli matrices are found as follows,

$$\sigma^1 = e + f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma^2 = -ie + if = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

$$\sigma^3 = h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Moreover, the parity operator P, the generator γ of the symmetry group $\mathbf{Z}_2$, operating on $\underline{\tilde{H}}$, acts on $\tilde{c}$, as multiplication by (-1), see WS. Therefore, it maps the basis $\{(c_1 + d_1), (c_2 + d_2)\}$ into $\{(-c_1 + d_1), (-c_2 + d_2)\}$. Consequently, we have an isomorphism,

$$P : B_o \to B_p,$$

Chirality

IX No 4

Chirality, in the physicists language, is explained as follows. The morphism P, extended to $B_o \oplus B_p$, in the basis chosen above, is given by the matrix,

$$P = \begin{pmatrix} 0 & id \\ id & 0 \end{pmatrix},$$

which turns left-handedness to right-handedness, with respect to the direction (o, p), resp. (p, o).

The Dirac Matrices

IX No 5

This shows that it is meaningful to consider the representations given by the Dirac matrices,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \ k = 1, 2, 3.$$

as well as the new operators,

$$\gamma^{k+3} = \begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix}, \ k = 1, 2, 3,$$

acting on, $B_o \oplus B_p$, such that,

$$\forall p \neq q, \ \gamma^p \gamma^q = -\gamma^q \gamma^p, \gamma^p \gamma^p = 1, \ p, q = 1, 2, 3, 4, 5, 6.$$

X The Generic Time-Action

No 1

Fix a metric g, for $\underline{\tilde{H}}$. Consider a representation,

$$\rho_0 : \tilde{H} \to End_k(\mathfrak{B}), \rho \in \tilde{H}/\bar{\mathfrak{g}}$$

and an extension $\rho_\psi : \tilde{H}(\sigma_g) \to End_k(\mathfrak{B})$, as a *momentum* of ρ_0, in t*he tangent direction* $\psi \in \mathbf{T}_{\rho_0} = End_{\tilde{H}}(\mathfrak{B})^n$, then

$$\rho_\psi(dt_i) = \xi_i + \psi_i, \ \psi_i \in End_{\tilde{H}}(\mathfrak{B})$$

corresponds to a ρ_0-derivation,

$$\eta \in Der_k(\tilde{H}, End_k(\mathfrak{B})), mappingto : 0 \in Ext^1_{\tilde{H}}(\mathfrak{B}, \mathfrak{B})$$

Then $ad(\rho_\psi(g - T)) = ad(Q_h + [\psi] + Q_v)$ where

$$[\psi] := \sum_i \psi_i \delta_{t_i}, \ Q_h = 1/2 \sum g_{i,j} \xi_i \xi_j, \ Q_v = 1/2 \sum g_{i,j} \psi_i \psi_j$$

The Generic Time-Action

X No 2

Time is then given by,

$$\delta = ad(\rho_\psi(g - T)) = ad(Q_h + [\psi] + Q_v)$$

meaning that,

$$\rho_\psi(d^{n+1}t_i) = [Q_h + [\psi] + Q_v, \rho_\psi(d^n t_i)].$$

This takes care of the complete time development of an operator in V, as well as a state vector $\phi \in \mathfrak{B}$.
The second order time-development is also given in terms of the Force Law valid in $\tilde{(}H)(\sigma_g)$,

$$d^2 t_i = -\sum_{p,q} \Gamma^i_{p,q} dt_p dt_q - \sum_{p,q} g_{p,q} F_{i,p} dt_q + 1/2 \sum_{l,p,q} g_{p,q}[F_{i,q}, dt_p]$$
$$+ 1/2 \sum_{l,p,q} g_{p,q}[dt_p, (\Gamma^{i,q}_l - \Gamma^{q,i}_l)]dt_l + [dt_i, T].$$ *Maxwell*

The Generic Time-Action

X No 3

Given the gauge group $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, there is a particular tangent direction, $\mathfrak{v} = \{\mathfrak{v}_i\}$, with $\mathfrak{v}_i = \sum_j \gamma_j g^{j,i}$, and $\gamma_i \in \mathfrak{g}_1$, "normal" to C in V, naturally related to a derivation, $\mathfrak{D} : \mathfrak{g}_0 \to Der_k(\mathfrak{g}_1)$. Put,

$$[\mathfrak{v}] := \sum_i \gamma_i \nabla_{\xi_i}, Q_h = 1/2 \sum g_{i,j} \xi_i \xi_j, \;\; Q_v = 1/2 \sum g_{i,j} \nu_i \nu_j$$

then with this choice of momentum, the time operator becomes,

$$\delta = ad(Q_h + [\mathfrak{v}] + Q_v) \in Der(End_k(\tilde{V}))$$

with the corresponding first order time-action in the state-space being defined by,

$$\delta = [\mathfrak{v}] + Q \in End_k(V).$$

and the curvature equal to,

$$F_{i,j} = ad([\mathfrak{v}_i, \mathfrak{v}_j]) \; \textit{Bloch}$$

XI Time and Furniture

No 1

Let $\mathbf{M}$ be the space of (isomorphism classes of) metrics on $\tilde{H}$. For every point $g \in \mathbf{M}$ consider the diagram,

$$\begin{tikzcd}
 & Ph(\tilde{H}) \arrow[r] \arrow[d] \arrow[dr, "\rho_1"] & Ph^2(\tilde{H}) \arrow[r] \arrow[d, "\rho_2"] & Ph^3(\tilde{H}) \ldots \arrow[r] \arrow[dl, "\rho_3"'] & Ph^\infty(\tilde{H}) \circlearrowleft \delta \\
\tilde{H} \arrow[ur, "d"] \arrow[r, "i"'] & \tilde{H}(\sigma_g) \arrow[r, "\rho"'] & End_k(\mathfrak{B}) & &
\end{tikzcd}$$

Where ρ_1 is the representation of $Ph(\tilde{H})$ induced by the representation ρ of $\tilde{H}(\sigma_g)$. Any extension ρ_2 of ρ_1 correspond to a tangent of ρ_1, therefore to an acceleration of $\rho_0 := i \circ \rho$. Let, $\mathbf{T}_{\mathbf{M},g}$ be the tangent space to $\mathbf{M}$, at g, i.e. $\mathbf{T}_{\mathbf{M},g} = \{(h_{i,j}\},\ h_{i,j} = h_{j,i} \in C)$. This implies that,

$$\mathbf{T}_{\mathbf{M},g} \subset Ext^1_{Ph(\tilde{H})}(\mathfrak{B}, \mathfrak{B})$$

More precisely, any first order deformation of the $Ph(\tilde{H})$-representation $(\rho_1, \mathfrak{B})$, induces a deformation of the metric g.

X General Relativity and Quantum Field Theory

No 1

Let us pick a basis, $\{\gamma_i\}$, $i = 1, ..., 6.$, in the finite dimensional Lie algebra $\mathfrak{g}$, and consider the operator,

$$[\delta] := \sum_{p,q} g^{p,q} \gamma_p \delta_{t_q} : End_k(\Theta) \to End_k(\Theta).$$

We may consider a family of representations,

$$\tilde{\rho} : Ph(\tilde{H}) \to End_{\tilde{H}}(\Theta)$$

defined by,

$$\tilde{\rho}(t_i) = t_i, \tilde{\rho}(dt_i) = \sum_j g^{i,j} \gamma_j.$$

Dynamics

The dynamics comes out as,

$$\tilde{\rho}(d(a)) = [\delta](\tilde{\rho}(a)) + [Q, \tilde{\rho}(a)].$$

where the Hamiltonian, $Q := \tilde{\rho}(g - T)$. Moreover, using the definition (1) above, we find after a short computation,

$$\tilde{\rho}(d^2 t_i) = [\delta](\tilde{\rho}(dt_i)) + [Q, \tilde{\rho}(dt_i)].$$

In fact this follows from the well known formula, $\delta_{t_q} g^{i,r} = -(\sum_l g^{i,l}\Gamma^r_{q,l} + g^{i,l}\Gamma^r_{q,l})$, using the physically equivalent to the Force Law of $\tilde{H}(\sigma_g)$, given by the formula (1) mentioned above,

$$d^2 t_i = -1/2 \sum_{p,q} (\Gamma^i_{p,q} + \bar{\Gamma}^i_{q,p}) dt_p dt_q \tag{5}$$

$$+ 1/2 \sum_{p,q} g_{p,q}(R_{p,i} dt_q + dt_p R_{q,i}) + [dt_i, T] \tag{6}$$

The Generalised Dirac Equation

No 19

The energy-mass-equation looks very much like an equations of Dirac type, and in fact, assuming the metric is trivial, and having picked constant gauge fields $\{\gamma_i\}$ properly, we find that $Q = \sum_i \gamma_i^2$ is the identity, and then Theorem (1.9) actually produces the Weyl equation, for Weyl-spinors,

$$[\delta](\xi) := \sum_i \sigma^i \delta_{t_i}(\xi) = E\xi,\ \xi \in B_o,$$

and the Dirac equation, for Dirac-spinors,

$$[\delta](\xi) := \sum_i \gamma^i \delta_{t_i}(\xi) = E\xi,\ \xi \in B_o, \oplus B_p.$$

But, beware, the parity operator,

$$P = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix},$$

does not preserve the Weyl spinors.

Quantum Klein-Gordon

Put,

$$\kappa := \sum_{p,q=1,2,3} g^{p,q}\gamma^p\delta_{t_q},\ \mu := \sum_{p,q=4,5,6} g^{p,q}\gamma^p\delta_{t_q},$$

and notice that since, by definition $\tilde{c}$ and $\tilde{\Delta}$ are normal, we have $\sum_p g^{p,q}g^{p,q'} = 0$, for $q \in \tilde{c}, q' \in \tilde{\Delta}$.
Then we find,

$$\{\kappa, \mu\} = 0,\ (\kappa+\mu)^2 = \sum_{i=1}^{6} \xi_i^2,\ \xi_i := \sum_{q=1}^{6} g^{i,q}\delta_{t_q}.$$

proving that, $[\kappa^2, \mu^2] = 0$, and $(\kappa+\mu)^2 = \kappa^2 + \mu^2$. So if ω is a nonsingular eigenvector for both κ and μ, then,

$$\kappa(\omega) = \sum_{p,q=1,2,3} g^{p,q}\gamma^p\delta_{t_q}(\omega) = K\omega,\ \mu(\omega) = \sum_{p,q=4,5,6} g^{p,q}\gamma^p\delta_{t_q}(\omega) = m\omega.$$

Laplace-Beltrami

Consequently,,

$$\sum_{i=1}^{6} \xi_i^2(\omega) = E^2\omega = (K^2 + m^2)\omega.$$

This is, in our model, the combined Einstein, Klein-Gordon and Dirac equation. Notice that,

$$\sum_p g_{i,j}\xi_i\xi_j = \sum_{p,q} g^{p,q}\delta_{t_p}\delta_{t_q},$$

is the Laplace-Beltrami operator.

Summing Up

No 20

Summarizing, we observe that our Toy Model has furnished,

- a Field Theory (including a Y-M model) for connections on $\tilde{H}$-bundles $\mathfrak{B}$, i.e. for representations,

$$\rho : Ph(\tilde{H})/(\sigma_g) =: \tilde{H}(\sigma_g) \to End_k(\mathfrak{B})$$

  with Dirac derivation for representations, $[\delta] = 0$ and Hamiltonian $Q = \rho(g - T)$.

- a Quantum Field Theory for gauge fields, i.e. for representations,

$$\rho : Ph(\tilde{H})/([dt_i, t_j]) =: Ph(\tilde{H})_{part} \to End_{\tilde{H}}(\Theta_{\tilde{H}})$$

  with Dirac derivation $[\delta] = \sum_{p,q} g^{p,q} \gamma_p \delta_{t_q}$ and Hamiltonian $Q := \tilde{\rho}(g - T)$.

Summing Up

Both of these models are compatible with,

- a General Relativistic model, for the scheme $Spec(Ph(\tilde{H})_{com})$, i.e. for representations,

$$\rho : \tilde{H}(\sigma_0) := Ph(\tilde{H})_{com} \to End_{\tilde{H}(\sigma_0)}(\tilde{H}(\sigma_0)) \simeq \tilde{H}(\sigma_0)$$

  with Dirac derivation, $[\delta] = \sum(-\Gamma^i)\delta_{\xi_i} + \xi_i\delta_{t_i}$, and Hamiltonian $Q = 0$,

Exterior Forces

The exterior force fields in physics are in our Field Theory Model representations like $\rho : \tilde{H}(\sigma_g) \to End_k(\mathfrak{B})$. The dynamics of ρ is given by the time derivative $\dot{\rho}$ of ρ, being defined by,

$$\dot{\rho}(dt_i) = \rho(d^2 t_i) = -\sum_{p,q} \Gamma^i_{p,q} \nabla_p \nabla_q - 1/2 \sum_{p,q} g_{p,q}(F_{i,p}\nabla_q + \nabla_p F_{i,q})$$
$$+ 1/2 \sum_{l,p,q} g_{p,q}[\nabla_p, (\Gamma^{i,q}_l - \Gamma^{q,i}_l)]\nabla_l + [\nabla_i, T].$$

For trivial metric (or for the Minkowski metric), the equation reduces to,

$$\dot{\rho}(dt_i) = \rho(d^2 t_i) = -\sum_p (F_{i,p}\nabla_p + 1/2\delta_{t_p}(F_{i,p})),$$

and the Hamiltonian $Q = ad(g - T) = \Delta$, reduces to the Laplace differential operator.

In this talk I shall sketch a mathematical model for a Big Bang scenario, based on relatively simple non commutative algebraic geometry.
Let U be the 4-dimensional real affine algebra of a point with a 3-dimensional tangent space, in $\mathbf{A}^3_{\mathbf{R}}$. The versal family of deformation of U, as associative $\mathbf{R}$-algebras, contains, in a natural way, the 6-dimensional moduli space, $\underline{H} := Hilb^2(\mathbf{R}^3)$, which is my Toy Model for the Big Bang-scenario in cosmology, introduced in (WS).
The study of the corresponding family of derivations of the 4-dimensional algebras, leads to a natural way of introducing an action of the gauge Lie algebras of the Standard Model, in $\Theta_{\underline{H}}$.
Introducing the notion of quotient spaces in non-commutative algebraic geometry, we obtain a geometry that seems to fit well with the set-up of the Standard Model.
These subjects are all treated within the set-up of (WS).

Standard Model

Fixing the Riemanian metric g on the space $\mathbf{H}$, we have the following structures uniquely defined,

- Fermions: $\tilde{\Delta}_{\mathbf{C}}$, with the action of $\mathfrak{su}(3)$
- Bosons: $\mathfrak{g}_{\mathbf{C}}$, with the regular representation of $\mathfrak{g}$

Ingredients of SM

- $\Theta_{\underline{\tilde{H}}} \simeq B_o \oplus B_p \oplus A_{o,p} \simeq \tilde{\Delta} \oplus \tilde{c}$; resp. 0-and light-velocities.
- $\mathfrak{u}(1) \simeq B_o \simeq B_p$.
- $\mathfrak{sl}(2) \subset \mathfrak{g} = Der_{\mathbf{H}}(\mathbf{U})$
- $\mathfrak{su}(3) := \mathfrak{su}_g(\tilde{\Delta}_{\mathbf{C}})$

The canonical action of these principal Lie bundles on the complexified tangent bundle $\Theta_{\mathbf{H}}$ gives us all Fields.

SUSY 1

Choose a point $BB \in \underline{\Delta}$, considered as the point $(0,0) \in \mathbf{A}^3 \times \mathbf{A}^3$, and treat $\underline{H}$ as a vector space. Recall that $\underline{\Delta}$, and therefore the point BB, are not contained in $\mathbf{H}$. Let, moreover $\mathbf{M}(BB)$ be the 4-dimensional sub-scheme of $\underline{H}$, consisting of the set $\{(o,p)\}$ where o and p are linear dependent. Consider the diagram,

$$\begin{tikzcd}
 & \mathfrak{g} \arrow[d] \arrow[dr, "\kappa"] & \\
\mathbf{M}(BB) \arrow[r] & \mathbf{H} & \Theta_{\mathbf{H}} & \tilde{\Delta} \arrow[l]
\end{tikzcd}$$

Theorem

- *There exists an H-linear map κ, defined by, $\kappa(\delta) = \sum_{i=1}^{3} \delta_i^0 (\frac{\partial}{\partial o_i} + \frac{\partial}{\partial p_i})$*
- *If o and p, are linear independent, i.e. if (o,p) is not in $\mathbf{M}(BB)$, then $\mathfrak{sl}(2) = \mathfrak{g}/ker(\kappa) \simeq \tilde{\Delta}$, as H-modules.*

SUSY 2

Define: $Q_i : \mathfrak{sl}(2)_{\mathbf{C}} \oplus \tilde{\Delta} \to \mathfrak{sl}(2)_{\mathbf{C}} \oplus \tilde{\Delta}$, $Q_1 = (\kappa, 0)$, $Q_2 = (0, \kappa^{-1})$ Then we have a N=1, SUSY: with $\{Q_1, Q_2\} = 1$, $[Q_i, P_j] = 0$
The graded Lie algebra $\mathbf{L} \subset End_{\mathbf{H}}(\mathfrak{sl}(2)_{\mathbf{C}} \oplus \tilde{\Delta})$, generated by Q_i and $\mathfrak{su}(3)$ and $\mathfrak{g}$, give us a SUSY-like duality between the Bosons and the Fermions of the model.
And one recognises the ingredients of the Standard Model, in the following canonical representations of, $\mathfrak{sl}(2) \subset \mathfrak{g} = Der_{\mathbf{H}}(\mathbf{U})$, and $\mathfrak{su}(3) := \mathfrak{su}_g(\tilde{\Delta}_{\mathbf{C}})$, respectively,

- $B_o \overset{P}{\simeq} B_p \subset \Theta_{\underline{\tilde{H}}}$; Weyl spinors.
- $B_o \oplus B_p \subset \Theta_{\underline{\tilde{H}}}$; Dirac spinors.
- $\tilde{\Delta} \subset \Theta_{\underline{\tilde{H}}}$; Gell-Mann, 8-fold way, colour.
- $d_3 \in \tilde{\Delta} \cap A_{o,p}$; up-quark
- $d_1, d_2 \in \tilde{\Delta}, < \ d_1, d_2 > \perp d_3$; left-right down-quark.

Consider the list of markers, or labels, $\mathfrak{h}_1$, $\mathfrak{h}_2$, $I_3^{\pm} = 3/4\mathfrak{h}_2 \pm 1/2\mathfrak{h}_1$, and $Y_W = 1/2\mathfrak{h}_2 \pm \mathfrak{h}_1$.

Markers	$1/2 \cdot h$	$\mathfrak{h}_1$	$\mathfrak{h}_2$	I_3	Y_W
d_1	$1/2$	$1/2$	$-1/3$	0	$-2/3$
d_2	$-1/2$	$-1/2$	$-1/3$	$-1/2$	$1/3$
d_3	0	0	$2/3$	$1/2$	$1/3$
$p_1 = d_1 d_3 d_3$	$1/2$	$1/2$	1	1	0
$p_2 = d_2 d_3 d_3$	$-1/2$	$-1/2$	1	1	0

Markers	$1/2 \cdot h$	$\mathfrak{h}_1$	$\mathfrak{h}_2$	I_3	Y_W
$n_1 = d_1 d_1 d_3$	1	1	0	$1/2$	-1
$n_2 = d_2 d_2 d_3$	-1	-1	0	$-1/2$	1
$n_{1,2} = d_1 d_2 d_3$	0	0	0	0	0
$e_L = d_1 d_1 d_1$	$3/2$	$3/2$	-1	0	-2
$e_R = d_1 d_1 d_2$	$1/2$	$1/2$	-1	$-1/2$	-1

Markers	$1/2 \cdot h$	$\mathfrak{h}_1$	$\mathfrak{h}_2$	I_3	Y_W
$W_1 = d_1 d_2 d_2$	$-1/2$	$-1/2$	-1	-1	0
$\nu_L = d_1^{-1} d_2$	-1	-1	0	$-1/2$	1

To go from left to right handedness comes out by just exchanging d_1 and d_2. Here one may see that, $e_L + \nu_L = e_R$, and one easily find reasons for the decays,

$$n \to p + e + \nu_e, p^+ \to e^+ + \pi^0 \to e^+ + 2\gamma, d_1 \to d_3 + w^-, p^+ = e^- \to n + \nu_e.$$

Notice also that we may, in an obvious way, identify ν_L with the element $e + f$ in one coordinate, swaping d_1, d_2.

Further consequences of the Force Laws, Maxwell

It is not difficult to see that the evolution equation,

$$\dot{\rho}(dt_i) = \rho(d^2 t_i) = -\sum_p (F_{i,p}\rho(d_p) + 1/2\nabla_{\delta_s}(F_{i,p})).$$

restricted to the subspace $M(B)$ contains Maxwells equation. Interpreting,

$C = M(B),\ v_p = \dot{t}_p := \nabla_p = \nabla_{\delta_p}(A_p)(\underline{t}) \in End_{M(B)}(\Theta),\ \dot{v}_p := E\cdot\rho(d^2 t_i)(\underline{t})$

we find that the observed electromagnetic fact, i.e. the Lorentz force law,

$$\dot{v}_i = -\sum_p F_{i,p} v_p$$

implies that the tangent to $\dot{\rho}$, given by the potential $\{(1/2\sum_p \nabla_{\delta_p}(F_{i,p})), i = 1, ..., 4\}$, must be zero, (or as physicists might say, *gauge*). This means, that there exist $\Phi \in End_{M(B)}(\Theta_{M(B)})$, such that,

$$\{1/2\sum_p \nabla_{\delta_p}(F_{i,p}), i = 1, ..., 4\} = \{\nabla_{\delta_i}(\Phi) + [1/2\sum_p \nabla_{\delta_p}(F_{i,p})), \Phi], i = 1, ., 4$$

Further consequences of the Force Laws, Maxwell

Assuming that the electromagnetic field potentials, A_i are *scalars*, the curvatures, $F_{i,p} = [\nabla_{\delta_p}, A_i] - [\nabla_{\delta_i}, A_p]$ are also scalars, so all commutators $[1/2 \sum_p [\nabla_{\delta_p}, F_{i,p}], \Phi]$ vanish, implying,

$$1/2 \sum_p \nabla_{\delta_p}(F_{i,p}) = \nabla_{\delta_i}(\Phi), i = 1, ..4.$$

Now, let us choose coordinates such that $t_4 =: t_0 = \lambda$ (the proper time for physicists) is the only 0-velocity coordinate of $M(B)$, and use the classical notations, Simple computation then shows, where ∇_s is the (light-) space part of ∇. From this, and from the vanishing of the tangent to $\dot{\rho}$, i.e. the equation above, the Maxwell equations follows, with $\Phi = \nabla \cdot A$, *electric current* $\mathfrak{J} = \nabla_s(\Phi)$, and *electric charge* $\rho = \nabla_{t_0}(\Phi)$. Together, $(\rho, \mathfrak{J})$ is a 0-tangent to the representation ρ, the first component in $\tilde{c}$, i.e. in space (or light) direction, and the second in $\tilde{\Delta}$, or in 0-velocity direction.

Conclusions and Apology

The physical interpretation of the mathematical models presented here, and in the literature referred to in the next slide, should not, at the moment, be considered as a serious proposition for a new physics. With respect to the Big Bang, for example, I think any existing model, must be approached with utmost care.
However, I think the idea of defining time as a metric on the moduli space of the iso-classes of the mathematical models used to specify/define the physical phenomenon of interest, may be of some interest. I therefore proposed to name the study of the geometry of reasonable moduli spaces of this kind:
Geometry of Time-Spaces.

XII Interaction

XII No 1

Assuming given a dynamical structure and a Dirac derivation δ, we have associated to any representation $\rho : A(\sigma) \to Hom_k(V)$ its unique extension, or force acting on V via,

$$\psi(a) : V \to V, \psi = \delta \circ \rho.$$

For gauge fields, given as above by a representation of $\tilde{H}$ of the form $\mathfrak{B}$, where $\mathfrak{B}$ is a representation of the principal Lie-bundle $\mathfrak{g} \oplus \mathfrak{su}(3)$ defined on $\underline{\tilde{H}}$, we have studied self-interactions of representations,

$$\rho : Ph(\tilde{H})(\sigma_g) \to End_{\tilde{H}}(\mathfrak{B}),$$

and the force fields created by the derivations $\tilde{H} \to End_k(\mathfrak{B})$ induced by ρ. However, there are, in this case, other forces to be considered. The $\tilde{H}$ module $\mathfrak{B}$ may be a free, and $\mathfrak{sl}(2)$ and $\mathfrak{su}(3)$ being semi-simple, do not have nontrivial extensions, but the $rad(\mathfrak{g})$ part of $\mathfrak{g} = \mathfrak{sl}(2) \oplus rad(\mathfrak{g})$, has non-trivial extension modules, and as we shall see, this give us a model for the *Weak Interaction*.

Interaction

XII No 2

Consider the action of $\mathfrak{g} = \mathfrak{sl}(2) \oplus rad(\mathfrak{g})$, on $\tilde{c}$ and $\tilde{\Delta}$ then there is an isomorphisme,

$$Ext^1_{\tilde{H}(\mathfrak{g})}(\Theta, \Theta) \simeq Ext^1_{\mathfrak{g}}(\Theta, \Theta) \simeq Ext^1_{rad(\mathfrak{g})}(\Theta, \Theta)$$

and,

$$Ext^1_{rad(\mathfrak{g})}(\Theta, \Theta) = Ext^1_{rad(\mathfrak{g})}(\tilde{c}, \tilde{c}) \oplus Ext^1_{rad(\mathfrak{g})}(\tilde{\Delta}, \tilde{\Delta}) \oplus Ext^1_{rad(\mathfrak{g})}(\tilde{c}, \tilde{\Delta}) \oplus Ext^1_{rad(\mathfrak{g})}($$

Moreover,

$$Ext^1_{rad(\mathfrak{g})}(\tilde{c}, \tilde{c}) \simeq Ext^1_{rad(\mathfrak{g})}(\tilde{\Delta}, \tilde{\Delta})$$

and, any $\psi \in Ext^1_{rad(\mathfrak{g})}(\tilde{\Delta}, \tilde{\Delta})$ is given by the derivation,

$$\psi : rad(\mathfrak{g}) \to End_{\tilde{H}}(\tilde{\Delta}, \tilde{\Delta})$$

in the bases, $\{d_1, d_2, d_3\}$ of $\tilde{\Delta}$, the quarks, and $\{u, r_1, r_2\}$ of $rad(\mathfrak{g})$, modulo the trivial derivations, given by an element $a := (a_{i,j}) \in End_{\tilde{H}}(\tilde{\Delta}, \tilde{\Delta})$.

Interaction

XII No 3

We find,

$$\psi(u) = \begin{pmatrix} \alpha_{1,1} & 0 & \alpha_{1,3} \\ 0 & \alpha_{2,2} & \alpha_{2,3} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} \end{pmatrix} mod \begin{pmatrix} 0 & 0 & -a_{1,3} \\ 0 & 0 & -a_{2,3} \\ a_{3,1} & a_{3,2} & 0 \end{pmatrix}$$

where, $\alpha_{1,1} = \alpha_{2,2} = \alpha_{3,3}$, and

$$\psi(r_1) = \begin{pmatrix} -\alpha_{1,3} & 0 & 0 \\ -\alpha_{2,3} & 0 & 0 \\ \alpha^1_{3,1} & \alpha^1_{3,2} & \alpha_{1,3} \end{pmatrix} mod \begin{pmatrix} a_{1,3} & 0 & 0 \\ a_{2,3} & 0 & 0 \\ a_{3,3} - a_{1,1} & -a_{1,2} & -a_{1,3} \end{pmatrix}$$

$$\psi(r_2) = \begin{pmatrix} 0 & -\alpha_{1,3} & 0 \\ 0 & -\alpha_{2,3} & 0 \\ \alpha^2_{3,1} & \alpha^2_{3,2} & \alpha_{2,3} \end{pmatrix} mod \begin{pmatrix} 0 & a_{1,3} & 0 \\ 0 & a_{2,3} & 0 \\ -a_{2,1} & a_{3,3} - a_{2,2} & -a_{2,3} \end{pmatrix}$$

for some matrix $(a_{i,j})$.

Interaction

XII No 4

Recall the structure of $rad(\mathfrak{g})$. We have $[u, r_i] = \pm r_i, [r_1, r_2] = 0$, and the actions on Θ, in the basis $\{c_1, c_2, c_3, d_1, d_2, d_3\}$, given by,

$$u = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$r_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad r_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Interaction

XII No 5

Now, $Ext^1_{rad(\mathfrak{g})}(\tilde{\Delta}, \tilde{\Delta}) = Der(rad(\mathfrak{g}), End_k(\tilde{\Delta}, \tilde{\Delta}))/Triv$. The rest is just computation. In case (1), choose $a = \alpha$.Then we find,

$$\psi(u) = \begin{pmatrix} \alpha_{1,1} & 0 & 0 \\ 0 & \alpha_{2,2} & 0 \\ 2\alpha_{3,1} & 2\alpha_{3,2} & \alpha_{3,3} \end{pmatrix}$$

where, $\alpha_{1,1} = \alpha_{2,2} = \alpha_{3,3}$, and

$$\psi(r_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha^1_{3,1} & \alpha^1_{3,2} & 0 \end{pmatrix}$$

$$\psi(r_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha^2_{3,1} & \alpha^2_{3,2} & 0 \end{pmatrix}$$

since $\alpha_{1,2} = \alpha_{2,1} = 0$.

Interaction

XII No 6

If, in case (2), we choose $a = -\alpha$, we find,

$$\psi(u) = \begin{pmatrix} \alpha_{1,1} & 0 & 2\alpha_{1,3} \\ 0 & \alpha_{2,2} & 2\alpha_{2,3} \\ 0 & 0 & \alpha_{3,3} \end{pmatrix}$$

where, $\alpha_{1,1} = \alpha_{2,2} = \alpha_{3,3}$, and

$$\psi(r_1) = \begin{pmatrix} -2\alpha_{1,3} & 0 & 0 \\ -2\alpha_{3,2} & 0 & 0 \\ \alpha^1_{3,1} & \alpha^1_{3,2} & 2\alpha_{1,3} \end{pmatrix}$$

$$\psi(r_2) = \begin{pmatrix} 0 & -2\alpha_{1,3} & 0 \\ 0 & -2\alpha_{2,3} & 0 \\ \alpha^2_{3,1} & \alpha^2_{3,2} & 2\alpha_{2,3} \end{pmatrix}$$

since $\alpha_{1,2} = \alpha_{2,1} = 0$.

Interaction

XII No 7

Case (1) can be interpreted as follows: $Z_0 := \psi(u) - \alpha id, \alpha = \alpha_{i,i}$ mediates a force, mapping $\{d_1, d_2\}$ to d_3, $W_- := \psi(r_1)$ and $W_+ := \psi(r_2)$ do the same, but do not permute (d_1, d_2). In case (2), Z_0 maps d_3 to $\{d_1, d_2\}$, and W_-, W_+ sends $\{d_1, d_2\}$ to d_3 and permute (d_1, d_2), i.e. they change the orientation, see (6.3), and the relationship to left-and right-handedness, that is, to chirality.
This seems to be very close to the weak interaction on quarks, mediated by the weak interaction bosons, Z, W_+, W_-, identified above with the Bosons $u, r_1, r_2 \in \mathfrak{g}$, respectively.

A Canonical Filtration

I No 22

Put

$$Ph^{(n)}(A) := im\ i_n \subseteq Ph^{\infty}(A)$$

The k-algebra $Ph^{\infty}(A)$ has a descending filtration of two-sided ideals, $\{\mathbf{F}_n\}_{0\leq n}$ given inductively by

$$\mathbf{F}_1 = Ph^{\infty}(A) \cdot im(\delta) \cdot Ph^{\infty}(A)$$

and

$$\delta\mathbf{F}_n \subseteq \mathbf{F}_{n+1}, \quad \mathbf{F}_{n_1}\mathbf{F}_{n_2}\ldots\mathbf{F}_{n_r} \subseteq \mathbf{F}_n, \quad n_1 + \ldots + n_r = n$$

such that the derivation δ induces derivations $\delta_n : \mathbf{F}_n \longrightarrow \mathbf{F}_{n+1}$. Using the canonical homomorphism $i_n : Ph^n(A) \longrightarrow Ph^{\infty}(A)$ we pull the filtration $\{\mathbf{F}_p\}_{0\leq p}$ back to $Ph^n(A)$, obtaining a filtration of each $Ph^n(A)$ with,

$$\mathbf{F}_1^n = Ph^n(A) \cdot im(\delta) \cdot Ph^n(A)$$

A Canonical Filtration

I No 23

And inductively,

$$\delta \mathbf{F}^n_p \subseteq \mathbf{F}^{n+1}_{p+1}, \quad \mathbf{F}^n_{p_1} \mathbf{F}^n_{p_2} \ldots \mathbf{F}^n_{p_r} \subseteq \mathbf{F}^n_p, \quad p_1 + \ldots + p_r = p.$$

The Algebras of Higher Differentials

I No 24

Let $\mathbf{D}(A) := \varprojlim_{n\geq 1} Ph^{\infty}(A)/\mathbf{F}_n$, the completion of $Ph^{\infty}(A)$ in the topology given by the filtration $\{\mathbf{F}_n\}_{0\leq n}$. The k-algebra $Ph^{\infty}(A)$ will be referred to as the k-algebra of higher differentials, and $\mathbf{D}(A)$ will be called the k-algebra of formalized higher differentials. Put

$$\mathbf{D}_n := \mathbf{D}_n(A) := Ph^{\infty}(A)/\mathbf{F}_{n+1}$$

Clearly δ defines a derivation on $\mathbf{D}(A)$, and an isomorphism of k-algebras

$$\epsilon := exp(\delta) : \mathbf{D}(A) \to \mathbf{D}(A)$$

and in particular, an algebra homomorphism,

$$\tilde{\eta} := exp(\delta) : A \to \mathbf{D}(A)$$

inducing the algebra homomorphisms

$$\tilde{\eta}_n : A \to \mathbf{D}_n(A)$$

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