Auslander-Reiten Theory of Representation-Directed Artinian Rings

(Auslander-Reiten-Theorie von darstellungsgerichteten artinschen Ringen)

Diplomarbeit

von

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Erklärung

Hiermit versichere ich, die vorliegende Arbeit selbständig verfaßt und nur die angegebenen Hilfsmittel verwendet zu haben.

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Contents

Appendix

1 Introduction

The study of almost split sequences, and their arrangement in the Auslander-Reiten quiver, has proven a powerful tool for understanding the structure of the module category of a finite dimensional algebra (see, for instance, the book of Auslander, Reiten and Smalø [2] or the Lecture Notes of Ringel [11]).

The module categories of representation-finite artinian rings also have almost split sequences (as shown by Zimmermann [14]; a proof along common lines will be given in 3.20). It is therefore a natural question to ask to which extent presently known results for finite dimensional algebra over fields can be generalized to other artinian rings.

In the introductory Sections 2 and 3 we recall some basic definitions, and the formal concept of *strict τ-categories* (= categories with almost split sequences) due to Iyama [9, 10].

Then, Auslander-Reiten quivers of representation-directed artinian rings will be studied in Sections 4 to 9. *Representation-directed* artinian rings are artinian rings A with the property that there is no cycle of nontrivial morphisms between indecomposable modules.

When trying to write down something similar to the Auslander-Reiten formula, and even before that point, one faces the problem that there is no field present, which could be used for dualizing. The only available alternative consists in using instead the endomorphism rings of indecomposable modules m and n, which are at least skewfields. However, this results in two different duals of $\mathrm{Hom}_A(m,n)$, which will be denoted by $\mathrm{Hom}_A(m,n)^{\mathrm{L}}$ and $\mathrm{Hom}_A(m,n)^{\mathrm{R}}$. They are defined as $\mathrm{Hom}_{\mathrm{End}(m)}(\mathrm{Hom}_A(m,n),\mathrm{End}(m))$ and $\mathrm{Hom}_{\mathrm{End}(n)}(\mathrm{Hom}_A(m,n),\mathrm{End}(n))$, respectively. Adopting standard notation, let τm, $\tau^- n$ be the left/right terms in the almost split sequences with m, n as right/left terms. Then the Auslander-Reiten formula for representation-directed artinian rings, which will be proven in 4.5, reads

$$\overline{\mathrm{Hom}}_A(n,\tau m)^{\mathrm{R}} \cong \mathrm{Ext}^1_A(m,n) \cong \underline{\mathrm{Hom}}_A(\tau^- n,m)^{\mathrm{L}}.$$

With the help of this formula, one can read off directly from the Auslander-Reiten quivers not only information about homomorphisms, but also about extensions. This way, it will be shown in 4.8 that sincere representation-directed artinian rings have global dimension at most two.

For later use, *species* are defined in Section 5 as finite sets of skewfields with associated bimodules. A species is *simply connected* if its underlying graph is a tree. Similarly, one can define a simply connected artinian ring to be an artinian ring where a certain underlying graph is a tree. In 5.3, 5.5 it is shown that simply connected species and simply connected hereditary artinian rings have the same representation theory. In 5.12 it is shown that indecomposable representations of a simply connected representation-finite species are uniquely determined by their dimension vectors.

Dowbor, Ringel and Simson [5] introduced *dimension sequences* in order to describe Auslander-Reiten quivers of hereditary artinian rings having exactly two projective indecomposables. This concept is discussed in Section 6. In particular, it is shown in 6.7 that a certain 2×2 matrix ring is representation finite if a vector giving the dimensions of certain vector spaces is in fact a dimension sequence. It was shown by Schofield [13] that there are such rings which have Auslander-Reiten quivers that do not occur in the representation theory of artinian algebras.

In Section 7, combinatorial aspects of simply connected representation-finite artinian rings are discussed. It turns out that the translation quivers to be considered — the existence of certain skewfields and bimodules taken for granted — are indeed Auslander-Reiten quivers of artinian rings. It is shown that shifting dimension sequences has only little influence on the representation theory of the associated artinian rings, a fact which allows to reduce the complexity of combinatorial experiments. Further, it will be shown in 7.7 that the indecomposable modules over a simply connected representation-finite artinian ring are uniquely determined by their dimension vectors, by giving a combinatorial argument which reduces the proof to the statement from 5.12.

Following Ringel, we call a module over an artinian ring A a *brick* if its endomorphism ring is a skewfield, and two bricks are called *orthogonal* if there are no homomorphisms between them. Then one can construct, for any finite set of pairwise orthogonal bricks, an abelian subcategory of the module category A-mod where these bricks are the simple objects. It is shown in 8.4 that, under certain conditions on the vanishing of Ext^1, the subcategory determined by two orthogonal bricks is the module category of a hereditary artinian ring. This will be used to prove 8.10, the highlight of Section 8: *A species is representation-finite if and only if its corresponding quiver, with an edge labeled by the length of the dimension sequence of the bimodule belonging to the edge, is a Coxeter diagram.* This theorem was stated by Dowbor, Ringel and Simson in [5], where it is remarked that the proof is rather technical, and details are left out.

In Section 9, the Auslander-Reiten quivers of representation finite hereditary artinian rings are studied in detail. As we have just seen, they correspond to the Coxeter diagrams. In 9.10, a canonical bijection between the positive roots associated with a Coxeter diagram and the isomorphism classes of indecomposable modules of the corresponding hereditary artinian ring is constructed.

Bongartz and Gabriel [3] introduced the concept of coverings for module categories of finite dimensional algebras over fields. In Sections 10 to 14, we will be concerned with coverings of module categories of representation-finite artinian rings. Section 10 contains elementary properties of coverings of strict τ-categories. In Section 11, we define the *fundamental group* G of a strict τ-category $\mathcal{C}$ in a similar way the fundamental group is defined in topology, and construct, for every subgroup of G, a covering of $\mathcal{C}$. Especially, we will find a universal covering (11.6). It will turn out that all coverings having certain sensible properties arise from our construction (11.4). Coverings of module categories of representation finite artinian rings will be studied in section 13. It will be shown that the universal covering of such a module category is the module category of a locally representation finite ring (13.4). Concluding, one could say that, in some sense, the module categories locally all look like the ones of simply-connected artinian rings. Therefore, coverings provide a method of transferring theorems for simply connected representation-finite artinian rings to the more general situation of arbitrary representation-finite artinian rings. This method will be applied to the Auslander-Reiten formula, proving that it holds for arbitrary representation-finite artinian rings which have a graded module category (14.2). It will also follow that dimension sequences still determine certain relations between pairs of indecomposable modules (14.1).

In Section 15, a concept for gluing and ungluing of modules is developed. Ungluing provides a way of comparing module categories of some representation-finite artinian rings to module categories of simply connected ones, which is in some aspects simpler than applying the theory of coverings. The disadvantage is that this method can only be applied if the original module category has a suitable shape. Gluing provides a method of "building" large representation-finite simply connected artinian rings.

Finally, Appendix B contains the source code of a C++ program, which was written to produce, according to the description in 7.1, output like the Auslander-Reiten quivers given in Appendix C.

I wish to thank W. Rump for giving me this exciting subject, which he motivated in many great lectures and seminars, for his support, and for our interesting discussions. Also I would like to thank M. Hertweck for his helpful advice and for examining this thesis.

2 Representation-finite artinian rings — Definitions

In this section a few standard definitions on rings and modules are given. Also the notation for the necessary subcategories of module categories is introduced.

For a ring R let R-Mod, R-mod, R-Proj, R-proj, R-Inj and R-inj be the categories of left R-modules, finitely presented left R-modules, projective, finitely generated projective, injective, and finitely presented injective left R-modules, respectively. Let Mod-R be the category of right R-modules with the obvious variations.

Most of the time we will only be interested in left modules and just call them modules. We will always write homomorphisms of modules (and morphisms in categories that can and will be specialized to module categories) on the right and compose them accordingly.

2.1 Definition. A ring A is called *artinian* if it satisfies the descending chain condition on left and on right ideals, i.e. if every chain $A \supseteq I_0 \supseteq I_1 \supseteq \cdots$ of left or right ideals gets stationary.

If A is an artinian ring, then A is also noetherian, hence then the finitely generated and the finitely presented A-modules coincide, and they have finite length (see [1]).

2.2 Definition. A module $M \neq 0$ is called *indecomposable*, if it does not admit a nontrivial decomposition into a direct sum, i.e. $M = M_1 \oplus M_2 \Rightarrow M_1 = 0 \vee M_2 = 0$.

R-Ind denotes the category of indecomposable R-modules.

A module M is called *simple*, if it has exactly two submodules (in that case these two submodules are M and $\{0\}$).

For an artinian ring A the finitely generated indecomposable A-modules have local endomorphism rings (Fitting's Lemma, [1]). Therefore any finitely generated A-module has a unique (up to isomorphism and permutation) decomposition into indecomposable modules. Also any finitely generated A-module has a projective cover.

2.3 Definition. An artinian ring A is called *representation-finite* if there is only a finite number of isomorphism classes of indecomposable finitely generated A-modules.

2.4 Definition. An artinian ring A is called *artinian algebra* if A is finitely generated as a module over its center.

For example any finite dimensional algebra over a field is an artinian algebra.

For artinian algebras part of the results presented here can be found in [2]. Most of them with different proofs which cannot be extended to the case of arbitrary artinian rings since they use an explicit construction of τ ($= D\operatorname{Tr}$ in that case and in the notation of [2]) which does not work in the more general case.

2.5 Definition. A artinian ring is called *graded* if, as additive abelian group, $A \cong \oplus \operatorname{Rad}^i A/\operatorname{Rad}^{i+1} A$ and the multiplication in A is induced by the multiplications $\frac{\operatorname{Rad}^i A}{\operatorname{Rad}^{i+1} A} \times \frac{\operatorname{Rad}^j A}{\operatorname{Rad}^{j+1} A} \longrightarrow \frac{\operatorname{Rad}^{i+j} A}{\operatorname{Rad}^{i+j+1} A}$.

2.6 Definition. An artinian ring A is called *simply connected* if the graph with vertices the isomorphism classes of indecomposable projective modules, and, for indecomposable projectives P and Q, an arrow $[P] \longrightarrow [Q]$ whenever there is a non-trivial homomorphism $P \longrightarrow \operatorname{Rad} Q/\operatorname{Rad}^2 Q$, is a tree.

3 Strict τ-categories

Iyama introduced the concept of strict τ-categories [9, 10]. In order to follow his approach some basic properties of modules over preadditive categories are discussed at the beginning of this section. Then one-sided almost split sequences are introduced, and it is shown that they correspond to minimal projective resolutions in the module category (3.13). Strict τ-categories will be defined as categories, where every object has a right and a left almost split sequence such that these sequences are almost split unless the object is projective or injective, and it will be shown that the module categories of representation finite artinian rings have this property.

3.1 Lemma (Yoneda). *Let $\mathcal{C}$ be a preadditive category, $\mathtt{F} : \mathcal{C} \longrightarrow \mathtt{Ab}$ an additive functor ($\mathtt{Ab}$ denotes the category of abelian groups), $M \in \mathcal{Ob}(\mathcal{C})$. Then $\eta : Nat(\mathcal{C}(M,-),\mathtt{F}) \longrightarrow \mathtt{F}(M)$, $\alpha \longmapsto (1_M)\alpha_M$ is an isomorphism of abelian groups.*

Proof. $\eta^{-1} : a \longmapsto \alpha^a, \quad \alpha^a_N : \varphi \longmapsto (a)(\mathtt{F}(\varphi))$ ☐

To avoid set-theoretical problems, we will assume any categories over which we want to consider modules to be *skeletally small*, that means that the isomorphism classes of objects form a set (compare [12] to see how this restriction can be avoided).

3.2 Definition. Let $\mathcal{C}$ be a preadditive category. A $\mathcal{C}$-module is a contravariant functor $M : \mathcal{C} \longrightarrow \mathtt{Ab}$.

Let $\mathcal{C}$-Mod be the category of all $\mathcal{C}$ modules with natural transformations as morphisms, Mod-$\mathcal{C} = \mathcal{C}^{op}$-Mod.

3.3 Example. A ring R can be understood as a preadditive category $\mathcal{R}$ with a single object. Then $\mathcal{R}$-Mod $= R$-Mod (the "normal" R-modules).

3.4 Lemma. *The module $\mathcal{C}(-,m)$ is projective in $\mathcal{C}$-Mod for any $m \in \mathcal{C}$.*

Proof. Given α and π in the following diagram we have to find a natural transformation β that makes the diagram commutative.

$$\begin{tikzcd}
 & \mathcal{C}(-,m) \arrow[dl, dashed, "\beta"'] \arrow[d, "\alpha"] \\
M \arrow[r, two heads, "\pi"] & N
\end{tikzcd}$$

By Yoneda's Lemma $\alpha = (a)(N(-))$ for an $a \in N(m)$. That is equivalent to saying $(\varphi)\alpha_n = (a)(N(\varphi)) \in N(n)$ for any $n \in \mathcal{Ob}\,\mathcal{C}$ and any $\varphi \in \mathcal{C}(n,m)$.

Choose $b \in M(m)$ with $(b)\pi_m = a$ (that is possible since π_m is epi). If $\beta = (b)(M(-))$ then for any $n \in \mathcal{O}b\,\mathcal{C}$ and $\varphi \in \mathcal{C}(n,m)$ one has

$$(\varphi)\beta_n\pi_n = ((b)M(\varphi))\pi_n = (b)\pi_m(N(\varphi)) = (a)(N(\varphi)) = (\varphi)\alpha_n.$$

Therefore $\beta\pi = \alpha$. $\square$

3.5 Definition. The category of *finitely generated* $\mathcal{C}$*-modules* is

$$\mathcal{C}\text{-Mod}_{\text{fg}} = \{M \in \mathcal{C}\text{-Mod} \mid \exists \text{ epi } \bigoplus_{\text{finite}} \mathcal{C}(-,m_i) \twoheadrightarrow M\}$$

The category of *finitely presented* $\mathcal{C}$*-modules* is

$$\begin{aligned} \mathcal{C}\text{-mod} = \{M \in \mathcal{C}\text{-Mod} \mid \exists \text{ projective resolution} \\ \bigoplus_{\text{finite}} \mathcal{C}(-,m_i) \longrightarrow \bigoplus_{\text{finite}} \mathcal{C}(-,n_i) \twoheadrightarrow M\} \end{aligned}$$

We shall define $\mathcal{C}$-proj $= \mathcal{C}$-Proj $\cap\, \mathcal{C}$-mod. (The full subcategory of projective objects in $\mathcal{C}$ will be denoted by $\text{Proj}(\mathcal{C})$; hopefully this will cause no confusion.)

Following [12], we call a category $\mathcal{C}$ *strongly noetherian* if $\mathcal{C}$-mod is abelian and noetherian, i.e. if $\mathcal{C}$-mod has kernels and satisfies the ascending chain condition on subobjects of any object.

Note that, if $\mathcal{C}$ is in fact an additive category, which means that $\mathcal{C}$ has a zero-object and biproducts, then the finite direct sums in the above definition can be omitted.

The following lemma shows that a category being strongly noetherian is very similar to a noetherian ring.

3.6 Lemma. *Let $\mathcal{C}$ be a preadditive category. Then the following are equivalent:*

(i) *$\mathcal{C}$ is strongly noetherian.*

(ii) *Any submodule of any finite direct sum of modules of the form $\mathcal{C}(-,m)$ is finitely generated.*

(iii) *Any submodule of any finitely generated $\mathcal{C}$-module is finitely generated.*

Proof. First assume $\mathcal{C}$ to be strongly noetherian, and U to be a submodule of the direct sum of the modules $\mathcal{C}(-,m_i)$ such that U is not finitely generated. Then there is an infinite chain

$$K_1 \lneq K_2 \lneq \cdots \leq U \leq \bigoplus \mathcal{C}(-,m_i)$$

with K_j finitely generated for any $j \in \mathbb{N}$. Now the quotients

$$(\bigoplus_i \mathcal{C}(-, m_i))/K_j$$

are finitely presented and therefore, since $\mathcal{C}$-mod is abelian, the kernels K_i are finitely presented. This contradicts the ascending chain condition.

Next assume any submodule of any finite direct sum of modules of the form $\mathcal{C}(-, m)$ is finitely generated and $U \leq M$ with M finitely generated. Then we have the following pullback diagram.

$$\begin{tikzcd} K \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\text{PB}"] & \bigoplus \mathcal{C}(-, m_i) \arrow[d] \\ U \arrow[r, tail] & M \end{tikzcd}$$

By assumption K is finitely generated and therefore U is finitely generated as well.

For the last implication assume that all submodules of finitely generated $\mathcal{C}$-modules are finitely generated. Then clearly $\mathcal{C}$-mod $= \mathcal{C}$-Mod$_{\text{fg}}$ and this category is abelian and noetherian. $\square$

3.7 Lemma. *Let $\mathcal{C}$ be an additive category such that idempotents split in $\mathcal{C}$. Then the functor* $\mathtt{F}: \mathcal{C} \longrightarrow \mathcal{C}$-proj $: m \longmapsto \mathcal{C}(-, m)$ *is an equivalence.*

Proof. Given $\psi \in \mathcal{C}(m, n)$, we can define $\mathtt{F}(\psi) : \mathcal{C}(-, m) \longrightarrow \mathcal{C}(-, n)$ by $(\varphi)(\mathtt{F}(\psi)_x) = \varphi\psi = (\psi)\mathcal{C}(\varphi, n)$. So $\mathtt{F}(\psi) = (\psi)\mathcal{C}(-, n)$. By Yoneda's Lemma every $\eta : \mathcal{C}(-, m) \longrightarrow \mathcal{C}(-, n)$ is of the form $\eta = (\psi)\mathcal{C}(-, n)$ for exactly one $\psi \in \mathcal{C}(m, n)$. Thus $\mathtt{F}$ is full and faithful.

Let $P \in \mathcal{C}$-proj. Then $\mathcal{C}(-, m) \twoheadrightarrow P$ for some $m \in \mathcal{C}$. Since P is projective this epimorphism splits. Since $\mathtt{F}$ is full, the corresponding idempotent in $\mathrm{End}(\mathcal{C}(-, m))$ corresponds to an idempotent in $\mathrm{End}_{\mathcal{C}}(m)$, which splits. $\square$

3.8 Example. Let $\mathcal{R} = \{R\}$ be as in Example 3.3. In $\mathcal{R}$ idempotents need not split. But $\mathcal{R}$-mod $=$ (add R)-mod, where add R is the full subcategory of R-mod of direct summands of direct sums of copies of R. The category add R is additive and in add R idempotents split. Therefore R-proj $= \mathcal{R}$-proj $=$ (add R)-proj $=$ add R.

3.9 Definition. An additive category $\mathcal{C}$ is called a *Krull-Schmidt* category if every object of $\mathcal{C}$ is isomorphic to a finite direct sum of objects with local endomorphism rings.

From now on $\mathcal{C}$ is assumed to be a Krull-Schmidt category.

3.10 Definition. The *radical* of $\mathcal{C}$ is the ideal $\mathcal{J}$ of $\mathcal{C}$ generated by the non-invertible morphisms between indecomposable objects. We denote by $\mathcal{J}^n$ the ideal generated by the compositions of any n elements of $\mathcal{J}$, and set $\mathcal{J}^{(n)} = \mathcal{J}^n/\mathcal{J}^{n+1}$.

3.11 Lemma. *Let c be an indecomposable object in $\mathcal{C}$. Then* $\mathrm{Rad}(\mathrm{End}(c)) = \mathcal{J}(c,c)$.

Proof. Clearly $\mathrm{Rad}(\mathrm{End}(c)) \subseteq \mathcal{J}(c,c)$ by definition of $\mathcal{J}$. Let $\varphi \in \mathcal{J}(c,c)$, and write $\varphi = \sum \alpha_i\psi_i\beta_i$ with $c \xrightarrow{\alpha_i} a_i \xrightarrow{\psi_i} b_i \xrightarrow{\beta_i} c$, the a_i, b_i being indecomposable and the ψ_i being non-isomorphisms. Then $\alpha_i\psi_i\beta_i$ cannot be invertible. Since $\mathrm{End}(c)$ is local, $\alpha_i\psi_i\beta_i \in \mathrm{Rad}(\mathrm{End}(c))$, and it follows that $\varphi \in \mathrm{Rad}(\mathrm{End}(c))$. $\square$

3.12 Definition. A sequence $c_1 \overset{\nu}{\rightarrowtail} c_2 \xrightarrow{\mu} c_3$ is called *right almost split sequence* if

1. $\nu \in \mathcal{J}(c_1, c_2)$ and $\mu \in \mathcal{J}(c_2, c_3)$
2. $\nu = \ker(\mu)$
3. μ has the *right factorization property*: every $\alpha \in \mathcal{J}(x, c_3)$, for some x, factors through μ:

$$\begin{tikzcd}
c_2 \arrow[r, two heads, "\mu"] & c_3 \\
 & x \arrow[u, "\alpha \in \mathcal{J}"'] \arrow[ul, dashed, "\exists"]
\end{tikzcd}$$

Notation: $P^c_d = \mathcal{C}(c,d)$ and $S^c_d = \mathcal{C}(c,d)/\mathcal{J}(c,d)$. (Note that $S^c_d = 0$ for $c \ncong d$ both indecomposable.) We set $P_d = P^-_d$ and $S_d = S^-_d$. Then the P_d are the projective $\mathcal{C}$-modules and the S_d are the corresponding semisimple ones.

3.13 Theorem. *The following are equivalent:*

(i) $c_1 \overset{\nu}{\rightarrowtail} c_2 \overset{\mu}{\twoheadrightarrow} c_3$ *is a right almost split sequence;*

(ii) $P_{c_1} \overset{P_\nu}{\rightarrowtail} P_{c_2} \xrightarrow{P_\mu} P_{c_3} \overset{\pi}{\twoheadrightarrow} S_{c_3}$ *is a minimal projective resolution of* S_{c_3} *in* $\mathcal{C}$-mod. *(Here π is the canonical projection.)*

Proof. We make the following observations:

Firstly ν and μ are in the radical of $\mathcal{C}$ if and only if P_ν and P_μ are in the radical of $\mathcal{C}$-mod.

Secondly,

$$\begin{aligned}\nu \text{ not mono} &\iff \exists\, 0 \neq \alpha :\ c_0 \longrightarrow c_1 :\ \alpha\nu = 0 \\ &\iff \exists\, 0 \neq P_\alpha :\ P_{c_0} \longrightarrow P_{c_1} :\ P_\alpha P_\nu = 0 \\ &\iff P_\nu \text{ not mono.}\end{aligned}$$

Thus assume that ν and P_ν are both mono. Then, note that exactness at P_{c_2} in (ii) means that

$$\forall \alpha :\ ((\alpha)P_\mu = 0 \iff \exists \beta :\ \alpha = (\beta)P_\nu).$$

Equivalently, $\forall \alpha :\ (\alpha\mu = 0 \iff \exists\beta :\ \alpha = \beta\nu)$ which means that $\nu = \ker(\mu)$.

Thirdly, $\mu \in \mathcal{J}(c_2, c_3) \iff \mathrm{Im}(P_\mu) \leq \mathcal{J}(-, c_3) \iff P_\mu \pi = 0$, and the sequence in (i) has the right factorization property if and only if $\forall \alpha \in \mathcal{J}(x, c_3) :\ \alpha = \beta\mu$, which is equivalent to $\forall \alpha \in \mathrm{Ker}(\pi) :\ \alpha \in \mathrm{Im}(P_\mu)$. $\square$

Thus right almost split sequences are determined uniquely (up to isomorphism) by their end term c_3.

Denote by $\tau c \rightarrowtail \vartheta c \longrightarrow c$ the right almost split sequence ending in c and by $c \longrightarrow \vartheta^- c \twoheadrightarrow \tau^- c$ the left almost split sequence beginning in c. These sequences will be called the right and left almost split sequences of c, respectively.

3.14 Definition. An *almost split sequence* is a left and right almost split sequence. Following [9, 10], a *strict τ-category* will be defined as a Krull-Schmidt category $\mathcal{C}$ where

1. every indecomposable $c \in \mathcal{O}b\,\mathcal{C}$ has a right almost split sequence which is either an almost split sequence or $\tau c = 0$;

2. every indecomposable $c \in \mathcal{O}b\,\mathcal{C}$ has a left almost split sequence which is either an almost split sequence or $\tau^- c = 0$.

3.15 Definition. An object p in a strict τ-category $\mathcal{C}$ is called *projective* if $\tau p = 0$ and *injective* if $\tau^- p = 0$.

3.16 Lemma. *Let $\mathcal{C}$ be a strict τ-category. Then the following hold:*

1. *An object p in $\mathcal{C}$ is projective if and only if every short exact sequence ending in p splits.*

2. *An object i in $\mathcal{C}$ is injective if and only if every short exact sequence beginning with i splits.*

Proof. In both cases one direction is trivial, since right (respectively left) almost split sequences cannot be split short exact sequences.

Assume p is projective and that $a \rightarrowtail b \twoheadrightarrow p$ is a non-split short exact sequence. Then the following diagram is commutative, where ψ exists by the factorization property of right almost split sequences, and φ is the kernel morphism.

$$\begin{tikzcd}
a \arrow[r, tail, "\alpha"'] \arrow[d, dashed, "\varphi"'] & b \arrow[r, two heads, "\beta"'] \arrow[d, dashed, "\psi"'] & p \arrow[d, equal] \\
0 \arrow[r, tail] & \vartheta p \arrow[r] & p
\end{tikzcd}$$

Now ψ factors through β since $\alpha\psi = 0$. But then the morphism in the right almost split sequence is a split epimorphism contradicting the definition.

The proof of the second point is dual. $\square$

3.17 Remark. For a strict τ-category $\mathcal{C}$ the maps τ and τ^- induce mutually inverse bijections between the isomorphism classes of indecomposable, non-projective and the isomorphism classes of indecomposable non-injective objects in $\mathcal{C}$. Especially, if $\mathcal{C}$ has, up to isomorphisms, only finitely many indecomposable objects, then the number of isomorphism classes of indecomposable projective objects is identical to the number of isomorphism classes of indecomposable injective objects.

3.18 Theorem. *Let $\mathcal{C}$ be an abelian Krull-Schmidt category such that any indecomposable $c \in \mathcal{O}b\,\mathcal{C}$ has a left and a right almost split sequence. Then $\mathcal{C}$ is a strict τ-category.*

Proof. Assume c indecomposable, non-projective. Then we have the following commutative diagram, where g exists by the factorization property of the left almost split sequence in the second row.

$$\begin{tikzcd}
\tau c \arrow[r, tail, "\nu"] \arrow[d, equal] & \vartheta c \arrow[r, two heads, "\mu"] & c \\
\tau c \arrow[r, tail, "\nu^-"] & \vartheta^-\tau c \arrow[r, two heads, "\mu^-"] \arrow[u, "g"] & \tau^-\tau c \arrow[u, "h"'] \\
& k \arrow[u, tail] \arrow[r, equal] & k \arrow[u, tail]
\end{tikzcd}$$

Note that ν^- is mono. The cokernel morphism h cannot factor through μ since ν^- doesn't split, so the right column splits. Then the middle column also splits and the component of ν^- which maps to k is 0, since it factors through μ^-. Therefore $k = 0$ and the two sequences are isomorphic.

Dually, if c is non-injective its left almost split sequence is an almost split sequence. □

3.19 Theorem. *Let $\mathcal{C}$ be an abelian Krull-Schmidt category. Assume that there are only finitely many indecomposable objects in $\mathcal{C}$, and that for any c, d among these $\mathcal{J}(c,d)$ is finitely generated as* $\mathrm{End}(c)$ *and as* $\mathrm{End}(d)$ *module. Then $\mathcal{C}$ is a strict τ-category.*

Proof. Let c be an indecomposable object of $\mathcal{C}$. For every indecomposable $d \in \mathcal{Ob}\,\mathcal{C}$ let $\{f_i^d : 1 \leq i \leq n_d\}$ be an $\mathrm{End}(d)$ generating set of $\mathcal{J}(c,d)$. Then $c \xrightarrow{(f_i^d)} \oplus_d d^{n_d} \twoheadrightarrow \mathrm{Cok}$ has the left factorization property with the left morphism being in $\mathcal{J}$.

Let $c \xrightarrow{\nu^-} \vartheta^-(c) \xrightarrow{\mu^-} \tau^- c$ be a sequence with these two properties in which the middle term has the smallest possible number of indecomposable direct summands. Then it is a left almost split sequence.

Dually c has a right almost split sequence.

Thus it follows from Theorem 3.18 that $\mathcal{C}$ is a strict τ-category. □

The most important examples of strict τ-categories in this thesis will be:

3.20 Theorem. *Let A be a representation-finite artinian ring. Then A*-mod *is a strict τ-category.*

Proof. Let c be a non-projective indecomposable A-module. Then there is a non-split short exact sequence $a \rightarrowtail b \twoheadrightarrow c$ with a indecomposable.

If possible take a non-invertible morphism $a \longrightarrow \tilde{a}$ with $\tilde{a}$ indecomposable such that the pushout of the sequence along this morphism is again non-split. Replace the sequence by the one obtained by the pushout and repeat this step. Since there are only finitely many indecomposable objects and each of them has finite length the composition of a sufficiently large number of non-invertible morphisms between them is always zero. Therefore one finally gets a sequence $a \rightarrowtail b \twoheadrightarrow c$ such that the pushout along any non-invertible morphism splits. That means that any non-invertible morphism $a \longrightarrow \tilde{a}$ must factor through b, so we found a left almost split sequence ending in c.

Dually, one can find, for any indecomposable non-injective c, a right almost split sequence beginning with c. Applying this to a one gets the following commutative diagram, where the first row is left almost split and the second one is right almost split. The center arrow exists by the factorization property of the upper sequence.

$$\begin{tikzcd} a \arrow[r, tail, "\alpha"] \arrow[d, equal] & b \arrow[r, two heads, "\beta"] \arrow[d, dashed] & c \arrow[d, dashed] \\ a \arrow[r, tail, "\gamma"] & d \arrow[r, two heads, "\delta"] & e \end{tikzcd}$$

By the factorization property of the lower sequence and the fact that the upper one is non-split it follows that the vertical arrows need to be isomorphisms.
If c is projective $0 \longrightarrow \operatorname{Rad} c \longrightarrow c$ is a right almost split sequence. If c is injective $c \longrightarrow c/\operatorname{Soc} c \longrightarrow 0$ is a left almost split sequence. $\square$

For any indecomposable object $c \in \mathcal{C}$ let $D(c) = \mathcal{J}^{(0)}(c,c)$ be the head of the endomorphism ring of c. Since $\mathcal{C}$ is Krull-Schmidt, this is a skewfield.

3.21 Theorem. *Let $\mathcal{C}$ be a strict τ-category and $c, d \in \mathcal{C}$ indecomposable. Then the following hold:*

1. *In a decomposition of ϑc into indecomposable direct summands there are exactly $\dim_{D(d)} \mathcal{J}^{(1)}(d,c)$ summands isomorphic to d.*

2. *In a decomposition of $\vartheta^- d$ into indecomposable direct summands there are exactly $\dim_{D(c)} \mathcal{J}^{(1)}(d,c)$ summands isomorphic to c.*

Especially, d is a direct summand of ϑc if and only if c is a direct summand of $\vartheta^- d$.

Proof. Assume $\dim_{D(d)} \mathcal{J}^{(1)}(d,c) = n$. Let $\{\varphi_1, \ldots, \varphi_n\} \subset \mathcal{J}^1(d,c)$ be representatives of a $D(d)$-basis of $\mathcal{J}^{(1)}(d,c)$. Then the dashed arrow in the following diagram exists by the right factorization property of the horizontal morphism.

$$\begin{tikzcd} & d^{(n)} \arrow[dl, dashed, "\begin{pmatrix}\psi_1\\ \vdots\\ \psi_n\end{pmatrix}"'] \arrow[d, "\begin{pmatrix}\varphi_1\\ \vdots\\ \varphi_n\end{pmatrix}"] \\ \vartheta c \arrow[r] & c \end{tikzcd}$$

The images of the ψ_i in $\mathcal{J}^{(0)}(d, \vartheta c)$ are linear independent, therefore $(\psi_i)_i$ is a split monomorphism and $d^{(n)}$ is a direct summand of ϑc.

Now assume that $d^{(n+1)}$ also is a direct summand of ϑc. Then there is a direct summand of ϑc isomorphic to d such that $\iota\mu \in \mathcal{J}^2$. (Here ι denotes the inclusion of the direct summand.) Then $\iota\mu = r\mu$ for some $r \in \mathcal{J}$ and, since ν is the kernel of μ, the map $\iota - r$ factors through ν as illustrated in the following diagram:

$$\begin{tikzcd} & d \arrow[dl, dashed] \arrow[d, "\iota - r"] & \\ \tau c \arrow[r, tail, "\nu"] & \vartheta c \arrow[r, "\mu"] & c \end{tikzcd}$$

But now $\iota - r \equiv \iota \not\equiv 0 \pmod{\mathcal{J}}$ is a split monomorphism which factors through ν, a contradiction, since $\nu \in \mathcal{J}$.

The second statement follows dually. □

3.22 Definition. Let $\mathcal{C}$ be a strict τ-category. The *Auslander-Reiten quiver* of $\mathcal{C}$ is the $\mathbb{Z}^2$-valued quiver with vertices the isomorphism classes of indecomposable objects in $\mathcal{C}$, and, whenever $\mathcal{J}^{(1)}(c,d) \neq 0$, an edge $c \longrightarrow d$ valued $(\dim_{D(c)} \mathcal{J}^{(1)}(c,d), \dim_{D(d)} \mathcal{J}^{(1)}(c,d))$.

Edges valued $(1,1)$ will be drawn as unvalued edges in order to simplify the pictures of the Auslander-Reiten quivers.

3.23 Definition. Let ${}_F M_G$ be bimodule over two skewfields F and G. Then the *left dual* of M is $M^{\mathrm{L}} = {}_G(\mathrm{Hom}_F(M,F))_F$ and the *right dual* of M is $M^{\mathrm{R}} = {}_G(\mathrm{Hom}_G(M,G))_F$.

If ${}_F M_G$ has finite F-dimension, then M is naturally isomorphic to M^{LR} as bimodule. Dually, if it has finite G-dimension, then $M \cong_{\mathrm{nat}} M^{RL}$.

3.24 Lemma. *Let $\mathcal{C}$ be a strict τ-category, $c \in \mathcal{O}b\,\mathcal{C}$ indecomposable and non-projective. Let $\tau c \rightarrowtail \vartheta c \twoheadrightarrow c$ be a fixed almost split sequence. Then this sequence induces an isomorphism of skewfields $D(\tau c) \cong D(c)$.*

Proof. Let $\varphi \in \mathcal{C}(c,c)$. We have the following commutative diagram, in which ψ exists by the right factorization property of the lower sequence:

$$\begin{tikzcd} \tau c \arrow[r, tail, "\nu"] \arrow[d, dashed, "\hat{\varphi}"'] & \vartheta c \arrow[r, two heads, "\mu"] \arrow[d, dashed, "\psi"'] & c \arrow[d, "\varphi"'] \\ \tau c \arrow[r, tail, "\nu"] & \vartheta c \arrow[r, two heads, "\mu"] & c \end{tikzcd}$$

We will show that $\varphi + \mathcal{J}(c,c) \longmapsto \hat{\varphi} + \mathcal{J}(\tau c, \tau c)$ with $\hat{\varphi}$ obtained as in the diagram is the desired isomorphism.

To see that the map is well-defined let $\varphi \in \mathcal{J}(c,c)$. Then φ factors through μ, and therefore $\hat{\varphi}$ factors through ν. So $\hat{\varphi} \in \mathcal{J}(\tau c, \tau c)$.

One can also apply the dual construction. Thus one also finds a map $D(\tau c) \longrightarrow D(c)$. Clearly these two maps are mutually inverse. It follows directly from the construction that the two maps are indeed ring homomorphisms. $\square$

3.25 Theorem. *Let $\mathcal{C}$ be a strict τ-category, $c \in \mathcal{Ob}\,\mathcal{C}$ indecomposable and non-projective. Let $\tau c \rightarrowtail \vartheta c \twoheadrightarrow c$ be a fixed almost split sequence and d an indecomposable direct summand of ϑc. Then there is an isomorphism of bimodules ${}_{D(d)}\mathcal{J}^{(1)}(d,c)^{\mathrm{L}}_{D(c)} \cong {}_{D(\tau c)}\mathcal{J}^{(1)}(\tau c, d)_{D(d)}$. (Here the identification $D(c) \cong D(\tau c)$ is induced by the almost split sequence as in Lemma 3.24.)*

Proof. First construct a map $\mathcal{J}^{(1)}(d,c) \otimes_{D(c)} \mathcal{J}^{(1)}(\tau c, d) \longrightarrow D(d)$: For $\psi \in \mathcal{J}(d,c)$ and $\varphi \in \mathcal{J}(\tau c, d)$ the factorization properties of the almost split sequence yield morphisms $\tilde{\psi}$ and $\tilde{\varphi}$ that make the following diagram commutative.

$$\begin{tikzcd}
& d & \\
\tau c \arrow[ur, "\varphi"] \arrow[r, tail, "\nu"] & \vartheta c \arrow[u, dotted, "\tilde{\varphi}"] \arrow[r, two heads, "\mu"] & c \\
& d \arrow[u, dotted, "\tilde{\psi}"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

To see that the map $(\psi + \mathcal{J}^2) \otimes (\varphi + \mathcal{J}^2) \longmapsto \tilde{\psi}\tilde{\varphi} + \mathcal{J}$ is well defined first assume $\varphi \in \mathcal{J}^2(\tau c, d)$. Then $\tilde{\varphi} \in \mathcal{J}(\vartheta c, d)$ and therefore $\tilde{\psi}\tilde{\varphi} \in \mathcal{J}(d,d)$. By the same argument $\psi \in \mathcal{J}^2$ implies $\tilde{\psi}\tilde{\varphi} \in \mathcal{J}(d,d)$. To see that the map is $D(c)$ balanced one must only remember the way $D(c)$ and $D(\tau c)$ are identified. With the morphisms as in the following commutative diagram the image of (the coset of) $(\psi\alpha, \varphi)$ is $(\tilde{\psi}\beta)\tilde{\varphi}$ and the image of $(\psi, \hat{\alpha}\varphi)$ is $\tilde{\psi}(\beta\tilde{\varphi})$.

$$\begin{tikzcd}
& d & \\
\tau c \arrow[ur, "\varphi"] \arrow[r, tail, "\nu"] & \vartheta c \arrow[u, dotted, "\tilde{\varphi}"] \arrow[r, two heads, "\mu"] & c \\
\tau c \arrow[u, "\hat{\alpha}"] \arrow[r, tail, "\nu"] & \vartheta c \arrow[u, "\beta"] \arrow[r, two heads, "\mu"] & c \arrow[u, "\alpha"'] \\
& d \arrow[u, dotted, "\tilde{\psi}"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

It follows directly from the construction that the map is $D(d)$ linear on both sides.

Now it is only necessary to check that the constructed map is non-degenerate in both entries. Assume $\varphi \in \mathcal{J}(\tau c, d) \setminus \mathcal{J}^2(\tau c, d)$. Then $\tilde{\varphi}$ cannot be in the radical and therefore is a split epimorphism. Choose $\tilde{\psi}$ such that $\tilde{\psi}\tilde{\varphi} = 1_d$ and $\psi = \tilde{\psi}\nu$. Then clearly $(\psi + \mathcal{J}^2) \otimes (\varphi + \mathcal{J}^2) \longmapsto 1$. Dually one can find, for any $\psi \in \mathcal{J}(d, c) \setminus \mathcal{J}^2(d, c)$, a $\varphi \in \mathcal{J}(\tau c, d)$ such that $(\psi + \mathcal{J}^2) \otimes (\varphi + \mathcal{J}^2) \longmapsto 1$. $\square$

4 Representation-directed artinian rings

In this section, we show that some known results on representation-directed artinian algebras generalize to artinian rings. In particular, a generalized version of the Auslander-Reiten formula will be established (4.5). With its help we will be able to prove that the following two assertions, which are well-known in case of representation-directed finite dimensional algebras over fields, remain true for arbitrary representation-directed artinian rings:

1. A module m has projective dimension at most one if and only if there are no homomorphisms from an injective module to τm (4.6).

2. If a sincere indecomposable module exists, then the global dimension of the ring is at most two (4.8).

4.1 Definition. A strict τ-category is called *directed* if, for any indecomposable $c_0, \ldots, c_n \in \mathcal{Ob}\,\mathcal{C}$ and non-isomorphisms φ_i:

$$c_0 \xrightarrow{\varphi_0} c_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{n-1}} c_n \xrightarrow{\varphi_n} c_0$$

at least one of the φ_i is 0. An artinian ring A is called *representation-directed* if A-mod is directed.

4.2 Lemma. *Let $\mathcal{C}$ be a directed strict τ-category, $c \in \mathcal{Ob}\,\mathcal{C}$ indecomposable. Then the endomorphism ring of c is a skewfield.*

Proof. By definition of directed, any non-isomorphism $c \longrightarrow c$ is zero. $\square$

4.3 Definition. For an artinian ring A we have the following sub- and quotient categories of A-mod:

1. The ideal of A-mod generated by the identities on projective modules, called $A\text{-mod}_{\mathcal{P}}$ or $[A\text{-proj}]$.

2. The ideal of A-mod generated by the identities on injective modules, called $A\text{-mod}_{\mathcal{I}}$ or $[A\text{-inj}]$.

3. The *stable category* $A\text{-}\underline{\text{mod}} = A\text{-mod}/A\text{-mod}_{\mathcal{P}}$.

4. The dual to the stable category, $A\text{-}\overline{\text{mod}} = A\text{-mod}/A\text{-mod}_{\mathcal{I}}$.

Notation: For $m, n \in A$-mod we write $\underline{\text{Hom}}(m,n) = \text{Hom}_{A\text{-}\underline{\text{mod}}}(m,n)$ and $\overline{\text{Hom}}(m,n) = \text{Hom}_{A\text{-}\overline{\text{mod}}}(m,n)$.

4.4 Lemma. *Let A be a representation-directed artinian ring, $m \in A$-mod indecomposable and non-projective and let $\tau m \rightarrowtail \vartheta m \twoheadrightarrow m$ be a fixed almost split sequence. Then this sequence induces an isomorphism of bimodules $D(m) \cong \operatorname{Ext}^1_A(m, \tau m)$.*

Proof. Let $\mathbb{E} \in \operatorname{Ext}^1(m, \tau m)$ be the fixed almost split sequence. Then the map $\Delta : D(m) \longrightarrow \operatorname{Ext}^1(m, \tau m) : d \longmapsto d\mathbb{E}$ is obviously a monomorphism of left vector spaces.

To see that it is also a morphism of right vector spaces remember that an endomorphism φ of m is turned into $\hat{\varphi} \in \operatorname{End}(\tau m)$ by use of the following diagram (Lemma 3.24).

$$\begin{tikzcd}
\tau m \arrow[r, tail, "\nu"] \arrow[d, dashed, "\hat{\varphi}"'] & \vartheta m \arrow[r, two heads, "\mu"] \arrow[d, dashed, "\psi"'] & m \arrow[d, "\varphi"'] \\
\tau m \arrow[r, tail, "\nu"] & \vartheta m \arrow[r, two heads, "\mu"] & m
\end{tikzcd}$$

Clearly, φ can be assumed to be nonzero and therefore invertible. Then $\hat{\varphi}$ is invertible as well. Now the following diagram is commutative.

$$\begin{tikzcd}
\mathbb{E} : & \tau m \arrow[r, tail, "\nu"] \arrow[d, "\hat{\varphi}"'] \arrow[dr, phantom, "\mathrm{PO}"] & \vartheta m \arrow[r, two heads, "\mu"] \arrow[d, equal] & m \arrow[d, equal] \\
\mathbb{E}\hat{\varphi} : \arrow[d, equal] & \tau m \arrow[r, tail, "\hat{\varphi}^{-1}\nu"] \arrow[d, equal] & \vartheta m \arrow[r, two heads, "\mu"] \arrow[d, "\psi"'] & m \arrow[d, equal] \\
\varphi\mathbb{E} : & \tau m \arrow[r, tail, "\nu"] \arrow[d, equal] & \vartheta m \arrow[r, two heads, "\mu\varphi^{-1}"] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{PB}"] & m \arrow[d, "\varphi"'] \\
\mathbb{E} : & \tau m \arrow[r, tail, "\nu"] & \vartheta m \arrow[r, two heads, "\mu"] & m
\end{tikzcd}$$

It follows that $\mathbb{E}\hat{\varphi} = \varphi\mathbb{E}$ in $\operatorname{Ext}^1(m, \tau m)$. Now $\Delta(d\varphi) = d\varphi\mathbb{E} = d\mathbb{E}\hat{\varphi} = \Delta(d)\hat{\varphi}$.

It remains to see that Δ is surjective. Let $\mathbb{F}$ be any non-split extension in $\operatorname{Ext}^1(m, \tau m)$. Then we get the following commutative diagram, where the center morphism exists by the left factorization property of the almost split sequence.

$$\begin{tikzcd}
\mathbb{F} : & \tau m \arrow[r, tail] & f \arrow[r, two heads] & m \\
\mathbb{E} : & \tau m \arrow[r, tail, "\nu"] \arrow[u, equal] & \vartheta m \arrow[r, two heads, "\mu"] \arrow[u, "\psi"] & m \arrow[u, "\varphi"]
\end{tikzcd}$$

Since the almost split sequence does not split φ is nonzero and therefore invertible. Hence $\mathbb{F} = \varphi^{-1}\mathbb{E} \in \Delta(D(m))$. $\square$

4.5 Theorem (Auslander-Reiten formula). *Let A be a representation-directed artinian ring, $m, n \in A\text{-ind}$. Then*

$$\overline{\mathrm{Hom}}(n, \tau m)^{\mathrm{R}} \cong \mathrm{Ext}^1(m, n) \cong \underline{\mathrm{Hom}}(\tau^- n, m)^{\mathrm{L}}$$

Proof. For m projective or n injective all three objects are clearly zero, so the formula holds in that case.

Let m be indecomposable, non-projective and fix an almost split sequence ending in m (or, equivalently, an isomorphism $\mathrm{Ext}^1(m, \tau m) \cong D(m)$). Then the composition

$$\Delta : \mathrm{Ext}^1(m, n) \otimes \mathrm{Hom}(n, \tau m) \longrightarrow \mathrm{Ext}^1(m, \tau m) \longrightarrow D(m)$$

is a homomorphism of bimodules.

Let $\varphi \in \mathrm{Hom}_{\mathcal{I}}(n, \tau m)$. That means there is an injective module i and there are morphisms $\varphi_1 : n \longrightarrow i$ and $\varphi_2 : i \longrightarrow \tau m$ such that $\varphi = \varphi_1\varphi_2$. Now for any $\mathbb{F} \in \mathrm{Ext}(m, n) : \Delta(\mathbb{F} \otimes \varphi) = \mathbb{F}\varphi_1\varphi_2 = 0$, since $\mathbb{F}\varphi_1 \in \mathrm{Ext}^1(m, i) = 0$. Therefore, Δ induces a morphism $\mathrm{Ext}^1(m, n) \otimes \overline{\mathrm{Hom}}(n, \tau m) \longrightarrow D(m)$ which will also be called Δ.

It remains to see that this new map Δ is non-degenerate in both entries.

First let $\mathbb{F} \in \mathrm{Ext}^1(m, n)$ be any non-split extension. By the right factorization property one can find a morphism ψ making the following diagram commutative.

$$\begin{tikzcd}
\mathbb{F}: & n \arrow[r, tail] \arrow[d, dashed, "\varphi"'] & f \arrow[r, two heads] \arrow[d, dashed, "\psi"'] & m \arrow[d, equal] \\
\mathbb{E}: & \tau m \arrow[r, tail, "\nu"] & \vartheta m \arrow[r, two heads, "\mu"] & m
\end{tikzcd}$$

But now the fixed almost split sequence is the pushout of $\mathbb{F}$ along φ and therefore $\Delta(\mathbb{F} \otimes \varphi) = 1$.

Now assume $\varphi \in \mathrm{Hom}(n, \tau m)$ such that $\varphi \neq 0$ in $A\text{-}\overline{\mathrm{mod}}$. Fix an injective coresolution $\mathbb{I} : n \rightarrowtail i \twoheadrightarrow \Omega^- n$ of n. The pushout $\mathbb{I}\varphi$ of $\mathbb{I}$ along φ cannot split since φ doesn't factor through i. Therefore the following diagram can be completed commutatively by the left factorization property of the almost split sequence.

$$\begin{tikzcd}
\mathbb{I}: & n \arrow[r, tail] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\mathrm{PO}"] & i \arrow[r, two heads] \arrow[d] & \Omega^- n \arrow[d, equal] \\
\mathbb{I}\varphi: & \tau m \arrow[r, tail] \arrow[d, equal] & e \arrow[r, two heads] & \Omega^- n \\
\psi\mathbb{I}\varphi = \mathbb{E}: & \tau m \arrow[r, tail] & \vartheta m \arrow[r, two heads] \arrow[u, dashed] & m \arrow[u, dashed, "\psi"]
\end{tikzcd}$$

Therefore, $\Delta(\psi\mathbb{I} \otimes \varphi) = 1$.

The first isomorphism of the theorem now follows, the second one is dual. $\square$

4.6 Corollary. *Let A be a representation-directed artinian ring, $m \in A$-ind. Then the following hold: (Here* pd *and* id *denote the projective and injective dimension, respectively.)*

1. $\operatorname{pd} m \leq 1 \iff \forall i \in A\text{-inj} : \operatorname{Hom}(i, \tau m) = 0$.

2. $\operatorname{id} m \leq 1 \iff \forall p \in A\text{-proj} : \operatorname{Hom}(\tau^{-} m, p) = 0$.

Proof. We will prove the first claim, the second one is dual.

Let n be any module and denote by $\Omega^{-} n$ its first cosyzygy module. Then $\operatorname{Ext}^2(m, n) = \operatorname{Ext}^1(m, \Omega^{-} n) = \overline{\operatorname{Hom}}(\Omega^{-} n, \tau m)^{\mathrm{R}}$.

Firstly, if $\operatorname{Hom}(i, \tau m) = 0$ for every injective i then $\operatorname{Hom}(\Omega^{-} n, \tau m) = 0$ for every n, since $\Omega^{-} n$ is the epimorphic image of an injective module. Therefore $\operatorname{Ext}^2(m, -) = 0$ which is equivalent to $\operatorname{pd} m \leq 1$.

Secondly, assume conversely that there is an injective module i and a nonzero morphism $\varphi : i \longrightarrow \tau m$. We may assume i to be indecomposable and i and φ chosen in such a way that φ does not factor through any other indecomposable injective module. That is possible since A is representation-directed. Clearly, φ cannot be a monomorphism, since then i would be a direct summand of τm. So the image v of φ is not injective and the induced morphism $v \longrightarrow \tau m$ does not factor through any injective module. By choosing $n = \ker(\varphi)$ we get $\Omega^{-} n = v$ and $\overline{\operatorname{Hom}}(\Omega^{-} n, \tau m) \neq 0$. By the equation above this is implies $\operatorname{pd} m \geq 2$. $\square$

4.7 Definition. Let A be an artinian ring. Following [11], a module $m \in A$-mod will be called *sincere* if every simple A-module is a composition factor of m. The artinian ring A will be called sincere if it has an indecomposable sincere module.

4.8 Corollary. *Let A be a representation-directed sincere artinian ring. Then the global dimension* $\operatorname{gld} A$ *is at most* 2.

Proof. Let m be a sincere indecomposable module and n any indecomposable module. We have to show that $\operatorname{pd} n \leq 2$, or, equivalently, that $\operatorname{pd} \Omega n \leq 1$, where Ωn denotes the first syzygy module of n. Assume this is false. Then by Corollary 4.6 there is an indecomposable injective module i and a nonzero morphism $i \longrightarrow \tau\Omega n$. Let s be an indecomposable direct summand of Ωn such that there is a nonzero morphism $i \longrightarrow \tau s$. Let d be a direct summand of ϑs and p an indecomposable direct summand of the projective cover of n such that $\mathrm{Hom}(s, p) \neq 0$. Then there are nonzero maps

$$i \longrightarrow \tau s \longrightarrow d \longrightarrow s \longrightarrow p \longrightarrow m \longrightarrow i.$$

The last two maps exist because m is sincere. This contradicts the definition of a representation-directed ring. $\square$

5 Species and hereditary artinian rings

The aim of this section is to understand how the module category of a hereditary artinian ring can be manipulated in such a way, that a simple projective representation is eliminated. This will be done in the category of representations of a species, so we first have to verify that these two categories are, for a suitable species, equivalent (5.5). Lastly we will see in 5.12 that successively eliminating projective objects gives us a proof that the indecomposable modules over a hereditary representation-finite artinian ring are uniquely determined by their dimension vectors.

5.1 Definition. An artinian ring is said to be *hereditary* if all submodules of projective modules are projective.

5.2 Definition. A *species* $\mathcal{S}$ is a finite set of skewfields $D_i, i \in I$ together with D_i-D_j-bimodules M_{ij}. All M_{ij} and their duals will be assumed to be finite-dimensional on both sides. A species will be called *ordered* if there is an ordering $\leq$ on I such that $M_{ij} = 0,\ \forall j \leq i$. For simplicity choose $I = \{1, \ldots, n\}$ with the natural order. A species is called *simply connected* if its underlying graph is a tree. (By the underlying graph we mean the graph with vertices I and arrows $i \longrightarrow j$ whenever $M_{ij} \neq 0$.) Clearly, a simply connected species is ordered.

A *representation* of a species $\mathcal{S}$ is a set of finite-dimensional vector spaces ${}_{D_i}V_i$ together with D_i linear maps $M_{ij} \otimes_{D_j} V_j \longrightarrow V_i$. We denote by $\mathrm{Rep}(\mathcal{S})$ the category of all representations of $\mathcal{S}$.

Notation: A species $\mathcal{S}$ will be written as a quiver with the skewfield D_i at the position of the i-th vertex and an arrow labeled M_{ij} from D_i to D_j whenever $M_{ij} \neq 0$.

An $\mathcal{S}$-representation will be written as the same quiver, but with the vector spaces V_i at the position of the i-th vertex and with unlabeled arrows. The maps $M_{ij} \otimes V_j \longrightarrow V_i$ will mostly be canonical, so they need not be explicitly given. Alternatively, if we do not know the underlying graph, we will also sometimes write an $\mathcal{S}$-representation as a tuple of vector spaces.

5.3 Lemma. *Let $\mathcal{S}$ be an ordered species. Then* $\mathrm{Rep}(\mathcal{S}) \approx A$-mod*, where A is the hereditary artinian ring*

$$\begin{pmatrix} D_1 & & \widetilde{M}_{ij} \\ & \ddots & \\ 0 & & D_n \end{pmatrix}, \quad \widetilde{M}_{ij} = \bigoplus_{i=i_0<i_1<\cdots<i_k=j} M_{i_0 i_1} \otimes M_{i_1 i_2} \otimes \cdots \otimes M_{i_{k-1} i_k}.$$

Proof. It follows directly from the definitions that the two categories are equivalent. Thus it remains to show that the artinian ring A is indeed hereditary. Clearly the indecomposable projective A-modules are exactly the columns of the matrix. Denote by P_n the projective module in the n-th column of the matrix. It is sufficient to show that the radical of any P_n is again projective.

$$\begin{aligned}
\operatorname{Rad} P_n &= \begin{pmatrix} \bigoplus\limits_{i=i_0<i_1<\cdots<i_k=n} M_{i_0 i_1} \otimes M_{i_1 i_2} \otimes \cdots \otimes M_{i_{k-1} i_k} \\ 0 \end{pmatrix}_i \\
&= \bigoplus_{j<n} \begin{pmatrix} \bigoplus\limits_{i=i_0<i_1<\cdots<i_k=j} M_{i_0 i_1} \otimes M_{i_1 i_2} \otimes \cdots \otimes M_{i_{k-1} i_k} \\ D_j \\ 0 \end{pmatrix}_i^{\dim_{D_j} M_{jn}} \\
&= \bigoplus_{j<n} P_j^{\dim_{D_j} M_{jn}}
\end{aligned}$$

Clearly this finite direct sum of projective modules is projective. $\square$

5.4 Corollary. *Let A be a graded hereditary artinian ring. Then A-mod $\approx$ $\operatorname{Rep}(\mathcal{S})$ for an ordered species $\mathcal{S}$.*

Proof. Let $\{P_i \mid 1 \leq i \leq n\}$ be the different indecomposable projective modules (up to isomorphism). Since A is hereditary we may assume that they are ordered in such a way that $\operatorname{Hom}(P_i, P_j) = 0$ whenever $i > j$. Then, for $D_i = \operatorname{End}(P_i)$ and $M_{ij} = \mathcal{J}^{(1)}(P_i, P_j)$, the ring of the lemma is Morita equivalent to A. $\square$

5.5 Corollary. *Let A be a simply connected hereditary artinian ring. Then A-mod $\approx \operatorname{Rep}(\mathcal{S})$ for a simply connected species $\mathcal{S}$.*

Proof. Since A is hereditary, for P and Q projective there is a non-trivial homomorphism $P \longrightarrow \operatorname{Rad} Q/\operatorname{Rad}^2 Q$ if and only if P is a direct summand of $\operatorname{Rad} Q$. As A is also simply connected, whenever $\mathcal{J}^{(1)}(P, Q) \neq 0$ for P and Q indecomposable projective we have $\mathcal{J}^{(1)}(P, Q) = \operatorname{Hom}(P, Q)$. Therefore A is graded and the first corollary can be applied. $\square$

5.6 Construction. Let $\mathcal{S}$ be a species with a vertex i such that for all vertices k the bimodule M_{ki} is zero. (That is equivalent to saying that i is a source of the underlying graph.) Assume $i = 1$. Let $V = (V_i)$ be a representation. Adding up the homomorphisms we get a map $\oplus_i M_{1i} \otimes V_i \longrightarrow V_1$. This map is surjective if V has no direct summands of the form $(D_1, 0, \ldots, 0)$.

Taking the kernel of this map one gets a map $\hat{V}_1 \longrightarrow \oplus_i M_{1i} \otimes V_i$ with components $\hat{V}_1 \longrightarrow M_{1i} \otimes V_i$. Now $M_{1i} \otimes V_i = M_{1i}^{RL} \otimes V_i = \operatorname{Hom}(M_{1i}^{\mathrm{R}}, D_i) \otimes V_i = \operatorname{Hom}(M_{1i}^{\mathrm{R}}, V_i)$, so we constructed maps $\hat{V}_1 \longrightarrow \operatorname{Hom}(M_{1i}^{\mathrm{R}}, V_i)$. Taking the adjoints we obtain maps $M_{1i}^{\mathrm{R}} \otimes \hat{V}_1 \longrightarrow V_i$. Thus we have constructed a representation of $\mathcal{S}'$, where $\mathcal{S}'$ is the species with the same skewfields as $\mathcal{S}$ and bimodules M'_{ij}, where $M'_{ij} = M_{ij}$ for $i \neq 1 \neq j$, $M'_{1i} = 0$ and $M'_{i1} = M_{1i}^{\mathrm{R}}$.

This construction yields a functor $\operatorname{Rep}(\mathcal{S}) \longrightarrow \operatorname{Rep}(\mathcal{S}')$. By applying the above steps inversely, we can also construct a functor $\operatorname{Rep}(\mathcal{S}') \longrightarrow \operatorname{Rep}(\mathcal{S})$, and these two functors induce mutually inverse equivalences between the subcategories $\{V \in \operatorname{Rep}(\mathcal{S}) \mid V \text{ has no direct summand } (D_1, 0, \ldots, 0)\}$ of $\operatorname{Rep}(\mathcal{S})$ and $\{V \in \operatorname{Rep}(\mathcal{S}') \mid V \text{ has no direct summand } (D_1, 0, \ldots, 0)\}$ of $\operatorname{Rep}(\mathcal{S}')$. Note that the $\mathcal{S}$ representation $(D_1, 0, \ldots, 0)$ is simple projective, while the $\mathcal{S}'$ representation $(D_1, 0, \ldots, 0)$ is simple injective.

The constructed functors are called *Coxeter functors* in [6, 7].

5.7 Definition. Let $\mathcal{S} = (D_i, M_{ij})$ be a species. Then we define the two species $\mathcal{S}^{\mathrm{R}}$ and $\mathcal{S}^{\mathrm{L}}$ as $\mathcal{S}^{\mathrm{R}} = (D_i, M_{ij}^{RR})$ and $\mathcal{S}^{\mathrm{L}} = (D_i, M_{ij}^{LL})$.

Note that, if there is a field k in the center of all D_i and operating centrally on the M_{ij}, such that, for all i $\dim_k D_i < \infty$, then $\mathcal{S} = \mathcal{S}^{\mathrm{L}} = \mathcal{S}^{\mathrm{R}}$. This follows, because the equality

$$\operatorname{Hom}(M_{ij}, D_i) \cong \operatorname{Hom}(M_{ij}, \operatorname{Hom}(D_i, k)) \cong \operatorname{Hom}(M_{ij}, k)$$

holds under these conditions. The rings corresponding to these species by Lemma 5.3 are finite dimensional k-algebras.

5.8 Theorem. *Let $\mathcal{S}$ be a simply connected species. Then there are functors $\hat{\tau} : \operatorname{Rep}\mathcal{S} \longrightarrow \operatorname{Rep}(\mathcal{S}^{\mathrm{R}})$ and $\hat{\tau}^- : \operatorname{Rep}\mathcal{S} \longrightarrow \operatorname{Rep}(\mathcal{S}^{\mathrm{L}})$ with*

1. $\hat{\tau}|_{\mathcal{S}\text{-proj}} = 0$
2. $\hat{\tau}^-|_{\mathcal{S}\text{-inj}} = 0$
3. *$\hat{\tau}$ and $\hat{\tau}^-$ induce mutually inverse equivalences between the $\mathcal{S}$ representations without projective direct summands and $\mathcal{S}^{\mathrm{R}}$ representations without injective direct summands.*

Proof. Since $\mathcal{S}$ is simply connected one can construct $\hat{\tau}$ by applying the Coxeter functors constructed in 5.6 for the different indices successively (see Example 5.10). ☐

Note that, in case the species corresponds to finite dimensional algebra over a field, $\hat{\tau}$ and τ coincide. This motivates to vary the definition of the Coxeter transformation (see [11]) analogously:

5.9 Definition. Let the *Coxeter transformation* be the linear map $\hat{\Phi}$ on the Grothendieck group defined by $\hat{\Phi} : \quad K_0[\mathcal{S}] \longrightarrow K_0[\mathcal{S}^{\mathrm{R}}]$, $\hat{\Phi}([P_i]) = -[I_i]$, where the P_i are the projective $\mathcal{S}$-representations and the I_i are the injective $\mathcal{S}^{\mathrm{R}}$-representations.

5.10 Example. Let $\mathcal{S}$ be the species $k \xrightarrow{k} k \xrightarrow{k} k$ for some field k. Then $\mathrm{Rep}(\mathcal{S})$ is a strict τ-category with Auslander-Reiten quiver Γ. Let Γ', Γ'' and Γ''' be the Auslander-Reiten quivers of the different species occurring "on the way to $\mathcal{S}^{\mathrm{R}}$". They are, together with the corresponding species, depicted in Table 1 on page 28, where the dashed vertical lines symbolize the identification by the Coxeter functor in 5.6.

For the remainder of this section assume that $\mathrm{Rep}(\mathcal{S})$ is a strict τ-category. By Theorem 3.20 this assumption is justified in case $\mathcal{S}$ is representation-finite.

5.11 Theorem. *Let $\mathcal{S}$ be a species. Let V be an $\mathcal{S}$-representation without projective direct summands. Then $[\hat{\tau}(V)] = \hat{\Phi}[V]$.*

Proof. The Coxeter functor constructed in 5.6 (call it $\mathtt{F}$ for the moment) induces a linear map f between the Grothendieck groups with $f([P_1]) = -[I_1]$ and $f([P_i]) = [P_i]$ for $i \neq 1$. Then $[\mathtt{F}(V)] = f[V]$ for all representations V without direct summands isomorphic to P_1. Iterating this the assertion follows. $\square$

An $\mathcal{S}$-representation R is called *preprojective* if, for some n, $\tau^n R = 0$ (or, equivalently, $\hat{\tau}^n R = 0$). We will denote the different preprojective $\mathcal{S}$ representations by $P^{\mathcal{S}}_{i,k} = (\tau^-)^k P^{\mathcal{S}}_i$. If $\mathcal{S}$ is representation-finite then every representation is of that form. Then, by the above theorem, $[P^{\mathcal{S}^{\mathrm{R}}}_{i,k-1}] = \hat{\Phi}[P^{\mathcal{S}}_{i,k}]$.

5.12 Theorem. *Let $\mathcal{S}$ be a simply connected species. Then any two non-isomorphic indecomposable preprojective representations have linearly independent dimension vectors.*

Proof. Assume $\alpha[P^{\mathcal{S}}_{i,k}] = \beta[P^{\mathcal{S}}_{j,l}]$ with $\alpha, \beta \in \mathbb{N} \setminus \{0\}$, $\mathcal{S}$ a simply connected species, i, j, k, l such that $\max\{k, l\}$ is minimal. It cannot happen that $k = l = 0$, since that would mean that two nonisomorphic projective modules have linearly dependent dimension vectors. If one of k and l is zero, for instance $k = 0$ and $l > 0$, then $\hat{\Phi}[P^{\mathcal{S}}_{i,k}] < 0$, while $\hat{\Phi}[P^{\mathcal{S}}_{j,l}] > 0$, a contradiction. If both k and l are positive, then $[P^{\mathcal{S}^{\mathrm{R}}}_{i,k-1}] = \hat{\Phi}[P^{\mathcal{S}}_{i,k}] = \hat{\Phi}[P^{\mathcal{S}}_{j,l}] = [P^{\mathcal{S}^{\mathrm{R}}}_{j,l-1}]$, which contradicts the minimaltity of $\max\{k, l\}$. $\square$

5.13 Corollary. *The indecomposable representations of a simply connected representation-finite species are uniquely determined by their dimension vectors.*

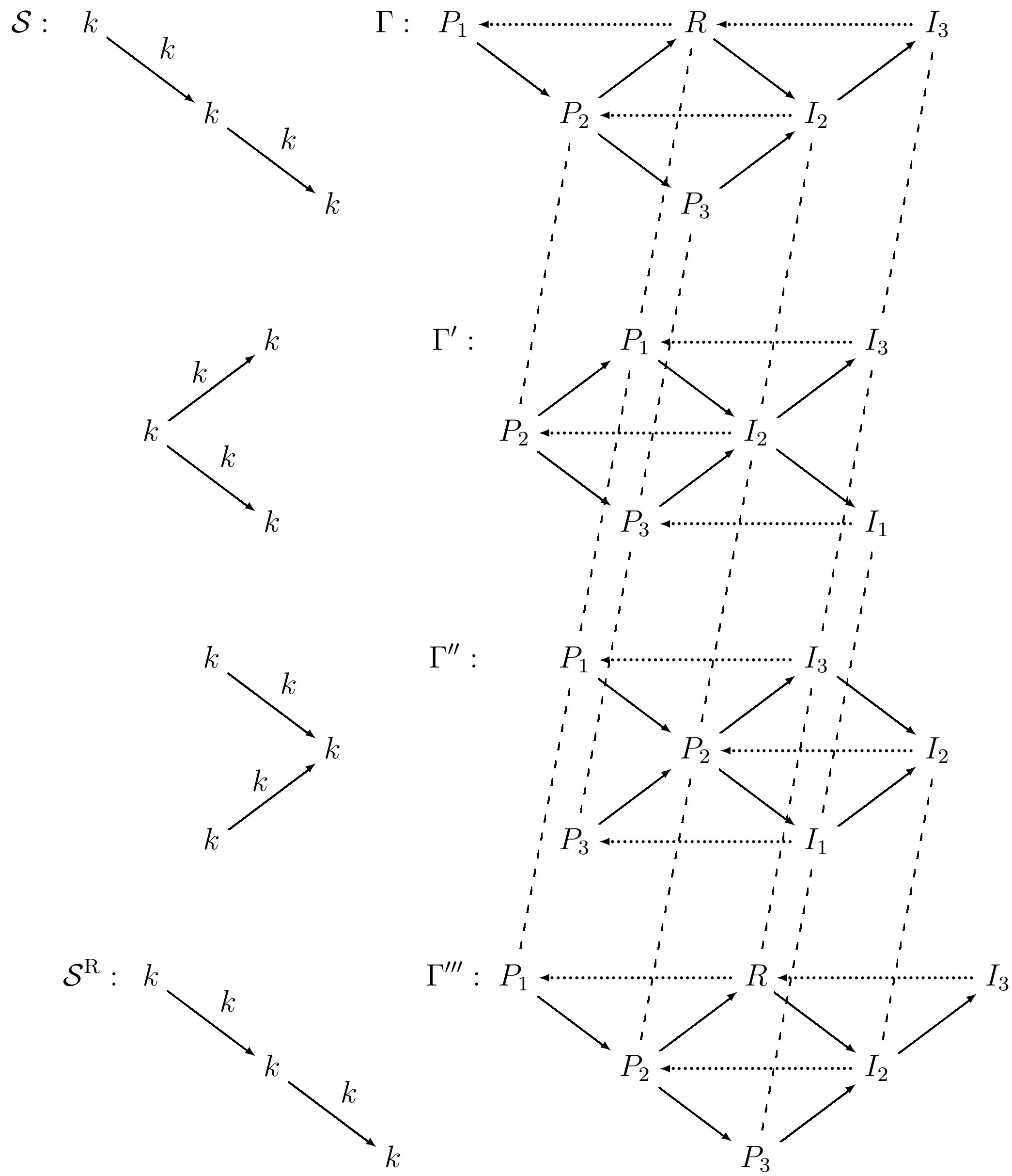


Table 1: Successive application of Coxeter functors

5.14 Corollary. *The indecomposable modules of a simply connected hereditary representation-finite artinian ring are uniquely determined by their dimension vectors.*

6 Dimension sequences — hereditary artinian rings with two projective modules

In [5] Dowbor, Ringel and Simson introduce dimension sequences, which they use to combinatorially classify the Auslander-Reiten quivers of hereditary artinian rings with exactly two indecomposable projective modules. In this section we will use their approach to show in 6.7 that the matrix ring $\left(\begin{smallmatrix} F & {}_FM_G \\ 0 & G \end{smallmatrix}\right)$, with M a bimodule over two skewfields F and G, is representation finite if and only if a sequence of dimensions of M and its duals is a dimension sequence. In order to do so, we will need to prove that the homomorphisms between projective modules are directly related (i.e. more than by taking duals through the entire Auslander-Reiten quiver) to the homomorphisms between the corresponding injective modules (6.6).

6.1 Definition. The set $\mathcal{D}$ of all finite *dimension sequences* is constructed as follows:

1. $(0,0)$ is a dimension sequence.
2. If $(d_1,\dots,d_n)$ is a dimension sequence then so is $(d_1,\dots,d_{i-1},d_i+1,1,d_{i+1}+1,d_{i+2},\dots d_n)$.

For a dimension sequence d its first entry will be called d_1, the second d_2 and so on continuing cyclically.

6.2 Lemma. *Let $(d_i)_{i=1\dots n}$ be a finite sequence of nonnegative integers. Then $(d_i)_{i=1\dots n}$ is a dimension sequence if and only if there are sequences $(x_i)_{i=0\dots n+1}$ and $(y_i)_{i=0\dots n+1}$ of integers such that the following properties hold.*

1. $x_0 = y_1 = x_n = y_{n+1} = 0,\ y_0 = x_{n+1} = -1,\ x_1 = y_n = 1.$
2. $x_i > 0,\ y_i > 0\ \forall i \in \{1\dots n\}.$
3. $d_i x_i = x_{i-1} + x_{i+1},\ d_i y_i = y_{i-1} + y_{i+1}\ \forall i \in \{1\dots n\}.$

Proof. The proof is done by induction on the length n of the sequence. For $n = 1$ there are no sequences either way. For $n = 2$ $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ can only be $\left(\begin{smallmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{smallmatrix}\right)$, and therefore $d = (0,0)$.

Assume $n > 2$ and d a dimension sequence of length n. Then there is a dimension sequence $\tilde{d}$ of length $n-1$ and an i such that

$$d = (\tilde{d}_1,\dots,\tilde{d}_{i-1},\tilde{d}_i+1,1,\tilde{d}_{i+1}+1,\tilde{d}_{i+2},\dots\tilde{d}_{n-1}).$$

Let $\tilde{x}, \tilde{y}$ be the corresponding sequences, which exist inductively. Then

$$x = (\tilde{x}_0, \dots, \tilde{x}_i, \tilde{x}_i + \tilde{x}_{i+1}, \tilde{x}_{i+1}, \dots, \tilde{x}_n) \text{ and}$$
$$y = (\tilde{y}_0, \dots, \tilde{y}_i, \tilde{y}_i + \tilde{y}_{i+1}, \tilde{y}_{i+1}, \dots, \tilde{y}_n)$$

satisfy the equations in the lemma.

Conversely, assume that such sequences x, y exists. Then it can be seen by induction on i that $\det\left(\begin{smallmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{smallmatrix}\right) = 1\ \forall i \in \{0 \dots n\}$. Therefore $\left(\begin{smallmatrix} x_i \\ y_i \end{smallmatrix}\right) \neq 0$. If $d(i) = 0$ for some i then $\left(\begin{smallmatrix} x_{i-1} \\ y_{i-1} \end{smallmatrix}\right) + \left(\begin{smallmatrix} x_{i+1} \\ y_{i+1} \end{smallmatrix}\right) = 0$. Hence one of these vectors has a negative entry, so i can only be 1 or n. If $d(1) = 0$ then $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ can only be $\left(\begin{smallmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & d_2 \end{smallmatrix}\right)$, therefore $n = 2$. Analogically $d(n)$ cannot be zero. Therefore $d(i) > 0\ \forall i$.

Assume, that $d(i) > 1\ \forall i$. Then the norms $|\left(\begin{smallmatrix} x_i \\ y_i \end{smallmatrix}\right)|$ grow monotonously, as $|\left(\begin{smallmatrix} x_i \\ y_i \end{smallmatrix}\right)| = |d_{i-1}\left(\begin{smallmatrix} x_{i-1} \\ y_{i-1} \end{smallmatrix}\right) - \left(\begin{smallmatrix} x_{i-2} \\ y_{i-2} \end{smallmatrix}\right)| \geq 2\,|\left(\begin{smallmatrix} x_{i-1} \\ y_{i-1} \end{smallmatrix}\right)| - |\left(\begin{smallmatrix} x_{i-2} \\ y_{i-2} \end{smallmatrix}\right)| \geq |\left(\begin{smallmatrix} x_{i-1} \\ y_{i-1} \end{smallmatrix}\right)|$, where the last inequality holds inductively. That contradicts $\left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$.

Therefore there is an i with $d_i = 1$. We set

$$\tilde{d} = (d_1, \dots, d_{i-1} - 1, d_{i+1} - 1, \dots, d_n),$$
$$\tilde{x} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \text{ and}$$
$$\tilde{y} = (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n).$$

These three sequences satisfy the conditions of the lemma, so, inductively, $\tilde{d}$ is a dimension sequence. Now the second point of the definition of dimension sequences says that d is a dimension sequence as well. □

6.3 Remark. The proof of the above lemma also shows that, for any dimension sequence and any "1" in the sequence, one can apply the inverse operation to the second point of the definition and thereby get a dimension sequence again.

6.4 Definition. Let $\sim$ be the equivalence relation on the set of all finite sequences generated by

1. $(x_i)_{i=1\dots n} \sim (x_i)_{i=1\dots kn}$
2. $(x_i)_{i=1\dots n} \sim (x_{i+1})_{i=1\dots n}$

We call two sequences $(x_i)_{i=1\dots n}$ and $(y_i)_{i=1\dots m}$ equivalent, if $(x_i)_{i=1\dots n} \sim (y_i)_{i=1\dots m}$.

6.5 Theorem. *Let A be an artinian ring. Let S_1, S_2 be simple A-modules, P_i and I_i the corresponding indecomposable projective and injective ones. Then the following holds:*

$$_{D(S_1)}\operatorname{Hom}(P_1, \operatorname{Rad} P_2/\operatorname{Rad}^2 P_2)_{D(S_2)}{}^{\mathrm{L}} \cong {}_{D(S_2)}\operatorname{Ext}(S_2, S_1)_{D(S_1)}$$
$$\cong {}_{D(S_1)}\operatorname{Hom}(\operatorname{Soc}(I_1/\operatorname{Soc} I_1), I_2)_{D(S_2)}{}^{\mathrm{R}}.$$

Proof. We will first prove the first isomorphism. In order for the assertion to make sense at all, one should first note that $D(P_1) = D(S_1)$ and that $\operatorname{Rad} P_2/\operatorname{Rad}^2 P_2$ is indeed a right $D(P_2)$-module.

Set $M = \operatorname{Rad} P_2/\operatorname{Rad}^2 P_2$ and $N = \operatorname{Tr}_M S_1$, where the trace $\operatorname{Tr}_M S_1$ of S_1 in M is the sum of all epimorphic images of S_1 in M. Let B be the component of the semisimple ring $A/\operatorname{Rad} A$ corresponding to S_1. (Then B is a full matrix ring over $D(S_1)$.) Now we clearly have a Morita equivalence ([1])

$$D(S_1)\text{-mod} \underset{\operatorname{Hom}_B(S_1,-)}{\overset{S_1 \otimes_{D(S_1)} -}{\rightleftarrows}} B\text{-mod}.$$

Therefore the equality between the second and third line of the following calculation holds.

$$\begin{aligned}
\operatorname{Hom}_A(P_1, M)^{\mathrm{L}} &= \operatorname{Hom}_A(S_1, M)^{\mathrm{L}} = \operatorname{Hom}_A(S_1, N)^{\mathrm{L}} = \operatorname{Hom}_B(S_1, N)^{\mathrm{L}} \\
&= \operatorname{Hom}_{D(S_1)}(\operatorname{Hom}_B(S_1, N), D(S_1)) \\
&= \operatorname{Hom}_B(N, S_1 \otimes D(S_1)) = \operatorname{Hom}_B(N, S_1) \\
&= \operatorname{Hom}_A(N, S_1) = \operatorname{Hom}_A(M, S_1)
\end{aligned}$$

Note that $\operatorname{Ext}^1_A(S_2, S_1) = \operatorname{Hom}_A(\operatorname{Rad} P_2, S_1)$, since no nonzero homomorphism $\operatorname{Rad} P_2 \longrightarrow S_1$ can factor through $\operatorname{Rad} P_2 \rightarrowtail P_2$. Therefore, and because S_1 is semisimple, $\operatorname{Ext}^1_A(\operatorname{Rad} P_2, S_1) = \operatorname{Hom}_A(M, S_1)$. The first isomorphism now follows.

In order to show the second isomorphism, one should note that, setting $M = \operatorname{Soc}(I_1/\operatorname{Soc} I_1)$, $N = \operatorname{Tr}_M S_2$ and B the component of S_2 in $A/\operatorname{Rad} A$, one gets a Morita duality (see [1])

$$\text{mod-}D(S_2) \underset{\operatorname{Hom}_B(-,S_2)}{\overset{\operatorname{Hom}_{D(S_2)}(-,S_2)}{\rightleftarrows}} (B\text{-mod})^{\mathrm{op}}.$$

The rest of the proof is analog to the proof of the first isomorphism. ☐

6.6 Corollary. *Let A be a hereditary artinian ring. Let S_i ($i \in \{1, 2\}$) be simple A modules and P_i and I_i the corresponding projective and injective ones. Then the following holds:*

$$\mathcal{J}^{(1)}(P_1, P_2)^{\mathrm{L}} \cong \operatorname{Ext}(S_2, S_1) \cong \mathcal{J}^{(1)}(I_1, I_2)^{\mathrm{R}}.$$

Notation: For a bimodule M over two skewfields we will write $M^{0\mathrm{L}} = M^{0\mathrm{R}} = M$, $M^{n\mathrm{L}} = (M^{(n-1)\mathrm{L}})^{\mathrm{L}}$, $M^{n\mathrm{R}} = (M^{(n-1)\mathrm{R}})^{\mathrm{R}}$, $M^{-n\mathrm{L}} = M^{n\mathrm{R}}$ and $M^{-n\mathrm{R}} = M^{n\mathrm{L}}$ for any positive n.

6.7 Theorem. *Let F and G be skewfields, M a bimodule. The artinian ring $\left(\begin{smallmatrix} F & {}_FM_G \\ 0 & G \end{smallmatrix}\right)$ is representation-finite if and only if there is an n such that $M^{n\mathrm{R}} \cong M$ as bimodules (this especially means $F \cong G$ in case n is odd), and the sequence $(\dim_G M, \dim_F M^{\mathrm{R}}, \dim_G M^{\mathrm{RR}}, \ldots, \dim_{\mathrm{left}} M^{(n-1)\mathrm{R}})$ is a dimension sequence.*

Proof. The ring $\left(\begin{smallmatrix} F & {}_FM_G \\ 0 & G \end{smallmatrix}\right)$ is representation-finite if and only if its Auslander-Reiten-Quiver is of the form

$$\begin{tikzcd}
\begin{pmatrix} F \\ 0 \end{pmatrix} \arrow[dr, "{(d_1,d_2)}"] & & \tau\begin{pmatrix} F \\ 0 \end{pmatrix} \arrow[ll, dotted] \arrow[dr, "{(d_3,d_4)}"] & & \cdots & \tau^-\begin{pmatrix} 0 \\ G \end{pmatrix} \arrow[dr, "{(d_{n-2},d_{n-1})}"] & & \begin{pmatrix} 0 \\ G \end{pmatrix} \arrow[ll, dotted] \\
& \begin{pmatrix} M \\ G \end{pmatrix} \arrow[ur, "{(d_2,d_3)}"] & & \tau\begin{pmatrix} M \\ G \end{pmatrix} \arrow[ll, dotted] & & & I_1 \arrow[ur, "{(d_{n-1},d_n)}"] &
\end{tikzcd}$$

Assume that there are (up to isomorphisms) n indecomposable modules. As $M = \mathrm{Hom}(P_1, P_2)$, Theorem 3.25 implies $\mathrm{Hom}(I_1, I_2)^{(n-2)\mathrm{L}}$ and, for any i, $d_i = \dim_{\mathrm{left}} M^{(i-1)\mathrm{R}}$. By Corollary 6.6 we have $\mathrm{Hom}(P_1, P_2) = \mathrm{Hom}(I_1, I_2)^{\mathrm{RR}}$, so $M \cong M^{n\mathrm{R}}$.

Let $\left(\begin{smallmatrix} x_i \\ y_i \end{smallmatrix}\right)$ be the dimension vector of the i-th module, $x_0 = y_{n+1} = 0$, $y_0 = x_{n+1} = -1$. Then x, y and d satisfy the conditions of Lemma 6.2 and therefore d is a dimension sequence.

Conversely, if the constructed sequence is a dimension sequence, then the dimension vectors of the modules in the Auslander-Reiten quiver can be calculated in the same way as the $\left(\begin{smallmatrix} x_i \\ y_i \end{smallmatrix}\right)$ are in Lemma 6.2. If a $(n+1)$-st module existed, its dimension vector would be negative, a contradiction. Therefore the ring is representation-finite. $\square$

Bimodules M with the property that an n exists such that $M^{n\mathrm{R}} \cong M$ and $(\dim_G M, \dim_F M^{\mathrm{R}}, \dim_G M^{\mathrm{RR}}, \ldots, \dim_{\mathrm{left}} M^{(n-1)\mathrm{R}})$ is a dimension sequence will be called dimension sequence bimodules.

6.8 Remark. Clearly bimodules with dimension sequence $(1,1,1)$, $(1,2,1,2)$ and $(1,3,1,3,1,3)$ exist (take for instance field extensions of degree 1, 2 and 3, respectively). It is shown in [13] that a dimension sequence bimodule with dimension sequence $(1,2,2,1,3)$ exists.

6.9 Remark. The existence of a dimension sequence bimodule with dimension sequence $(1,2,2,1,3)$ especially shows that there is in general, even for a hereditary representation-finite artinian ring A, no equivalence between A-proj and A-inj. Such an equivalence always exists for an artinian algebra A.

6.10 Lemma. *If ${}_FM_D$ and ${}_DN_G$ are bimodules over skewfields D, F and G, such that* $\dim_D N < \infty$ *or* $\dim_D M^{\mathrm{L}} < \infty$*, then* $(M \otimes N)^{\mathrm{L}} = N^{\mathrm{L}} \otimes M^{\mathrm{L}}$.

Proof. We have $\mathrm{Hom}(N, M^{\mathrm{L}}) = \mathrm{Hom}(N, D) \otimes M^{\mathrm{L}}$ since at least one of N and M^{L} is a finite dimensional vector space over D, so

$$\begin{aligned}(M \otimes N)^{\mathrm{L}} &= \mathrm{Hom}(M \otimes N, F) = \mathrm{Hom}(N, \mathrm{Hom}(M, F)) = \mathrm{Hom}(N, M^{\mathrm{L}}) \\ &= \mathrm{Hom}(N, D) \otimes M^{\mathrm{L}} = N^{\mathrm{L}} \otimes M^{\mathrm{L}},\end{aligned}$$

proving the lemma. □

6.11 Lemma. *Let ${}_FM_D$ and ${}_DN_G$ be bimodules over the skewfields D, F and G. If $\left(\begin{smallmatrix} F & {}_FM\otimes_D N_G \\ 0 & G \end{smallmatrix}\right)$ is representation-finite and $M \neq 0$, then $\left(\begin{smallmatrix} D & {}_DN_G \\ 0 & G \end{smallmatrix}\right)$ is representation finite as well.*

Proof. Assume that $\left(\begin{smallmatrix} D & {}_DN_G \\ 0 & G \end{smallmatrix}\right)$ is not representation-finite. Let m_i and n_i be $\dim_{\mathrm{left}} M^{(i-1)\mathrm{R}}$ and $\dim_{\mathrm{left}} N^{(i-1)\mathrm{R}}$, respectively. Then, by Lemma 6.10, the dimensions $\dim_{\mathrm{left}}(M \otimes N)^{(i-1)\mathrm{R}}$ must be $m_i n_i$. Let l_i be the length of the i-th module in the Auslander-Reiten quiver of $\left(\begin{smallmatrix} F & {}_FM\otimes_D N_G \\ 0 & G \end{smallmatrix}\right)$ and k_i the length of the i-th module in the Auslander-Reiten quiver of $\left(\begin{smallmatrix} D & {}_DN_G \\ 0 & G \end{smallmatrix}\right)$. By the assumption $k_i > 0$ for all positive i. Let $q_i = l_i/k_i$. Then

$$\begin{aligned}q_{i+1} &= \frac{l_{i+1}}{k_{i+1}} = \frac{m_i n_i l_i - l_{i-1}}{n_i k_i - k_{i-1}} = \frac{m_i n_i q_i k_i - q_{i-1} k_{i-1}}{n_i k_i - k_{i-1}} \\ &= m_i q_i + \frac{(m_i q_i - q_{i-1}) k_{i-1}}{k_{i+1}}\end{aligned}$$

Therefore, one can inductively see that the sequence $(q_i)_i$ grows monotonously. This clearly implies $l_i > 0$ for all positive i, so $\left(\begin{smallmatrix} F & {}_FM\otimes_D N_G \\ 0 & G \end{smallmatrix}\right)$ is also representation infinite. □

7 Combinatorial aspects of simply connected representation-finite artinian rings

In this section, it is first shown that, assuming the necessary skewfields and bimodules exist, the translation quivers coming from a combinatorial construction (7.1) are exactly the ones which are Auslander-Reiten quivers of some simply connected artinian ring. Next the complexity of our experiments with this construction is reduced in 7.6 by showing that cyclically shifting all dimension sequences has only little influence on the representation theory of the corresponding artinian ring and especially does not alter the number of indecomposable modules. Finally, it will be proved in 7.7 that the indecomposable modules of a simply connected representation-finite artinian ring are uniquely determined by their dimension vectors, by combinatorially reducing the question to the case of hereditary rings which has already been treated in 5.12.

In the following construction, we will build a translation quiver Γ. We will start with a finite quiver Q. The vertices of Q are to become the τ-orbits of Γ and the arrows in Q represent arrows between these τ-orbits. Since, in the Auslander-Reiten quiver of an artinian ring, the values on the arrows cyclically run through a dimension sequence, we value each arrow in Q with a dimension sequence. In order to also get the Auslander-Reiten quivers of non-hereditary artinian rings, we have to introduce a "start function" s, which tells every τ-orbit at which position its projective vertex is to be placed. Lastly, a length function l is needed, which is calculated in the same way as the lengths of modules calculate in an Auslander-Reiten quiver, to know when a τ-orbit is complete.

7.1 Construction. Let Q be a finite quiver with vertices Q_0 and edges Q_1. Let $d : Q_1 \longrightarrow \mathcal{D}\backslash\{(0,0)\}$ ($(0,0)$ is represented by "no arrow") be a function associating to each arrow a dimension sequence and $s : Q_0 \longrightarrow \mathbb{N}$ such that $s(j) - s(i) \in 2\mathbb{N}+1$ whenever there is an arrow from i to j.

We will define inductively a length function $l : Q_0 \times \mathbb{N} \longrightarrow \mathbb{N}$. Then we can define the translation quiver $\Gamma = (\Gamma_0, \Gamma_1, \tau)$ by

$$\Gamma_0 = \{(i,n) \in Q_0 \times \mathbb{N} \mid l(i,n) > 0\}$$

$$\Gamma_1 = \{(i,n-1) \xrightarrow{(d_1,d_2)} (j,n) \mid l(i,n-1) > 0, l(j,n) > 0 \text{ and } [(1) \text{ or } (2)]\}$$

with

$$(1) : (i \longrightarrow j) \in Q_1 \text{ and } (d_1,d_2) = (d(i \longrightarrow j)_{n-s(i)}, d(i \longrightarrow j)_{n-s(i)+1})$$

$$(2) : (j \longrightarrow i) \in Q_1 \text{ and } (d_1,d_2) = (d(j \longrightarrow i)_{n-s(j)}, d(j \longrightarrow i)_{n-s(j)+1})$$

$\tau(i,n) = (i, n-2)$ if $(i, n-2) \in \Gamma_0$

For $m \in \Gamma_0$ we will call $l(m)$ the length of m.

The function l is constructed inductively by the following rules:

1. $l(i,n) = 0$ if $n < s(i)$
2. $l(i,n) = (\sum_{(j \longrightarrow i) \in Q_1} d(j \longrightarrow i)_0 \cdot l(j, n-1)) + 1$ if $n = s(i)$
3. $l(i,n) = 0$ if $n > s(i)$ and $l(i, n-2) = 0$
4. $l(i,n) = \min\{0, \sum_{(j \longrightarrow i) \in Q_1} d(j \longrightarrow i)_{n-s(i)} \cdot l(j, n-1)$ $+ \sum_{(i \longrightarrow j) \in Q_1} d(i \longrightarrow j)_{n-s(j)} \cdot l(j, n-1)$ $-l(i, n-2)\}$ if $n > s(i)$ and $l(i, n-2) > 0$

These make our length function behave like the length of modules calculate in an Auslander-Reiten quiver.

7.2 Example. Let Q be the quiver $1 \longrightarrow 2 \longrightarrow 3$. Let $d: Q_1 \longrightarrow \mathcal{D}$ be defined by $d(1 \longrightarrow 2) = (1,2,2,1,3)$ and $d(2 \longrightarrow 3) = (1,1,1)$ and $s: Q_0 \longrightarrow \mathbb{N}$ by $s(1) = 1$, $s(2) = 2$ and $s(3) = 5$. Then all conditions of Construction 7.1 are met. The translation quiver obtained by the construction then is the following:

$$\begin{tikzcd}[column sep=small]
\begin{smallmatrix}(1,1)\\1\end{smallmatrix} \arrow[dr, "{(1,2)}"] & & \begin{smallmatrix}(1,3)\\3\end{smallmatrix} \arrow[ll, dotted] \arrow[dr, "{(2,1)}"] & & \begin{smallmatrix}(1,5)\\1\end{smallmatrix} \arrow[ll, dotted] \arrow[dr, "{(3,1)}"] & & \begin{smallmatrix}(1,7)\\3\end{smallmatrix} \arrow[ll, dotted] \arrow[dr, "{(2,2)}"] & & \begin{smallmatrix}(1,9)\\1\end{smallmatrix} \arrow[ll, dotted] \\
& \begin{smallmatrix}(2,2)\\2\end{smallmatrix} \arrow[ur, "{(2,2)}"'] & & \begin{smallmatrix}(2,4)\\4\end{smallmatrix} \arrow[ll, dotted] \arrow[ur, "{(1,3)}"'] \arrow[dr, "{(1,1)}"] & & \begin{smallmatrix}(2,6)\\4\end{smallmatrix} \arrow[ll, dotted] \arrow[ur, "{(1,2)}"'] & & \begin{smallmatrix}(2,8)\\2\end{smallmatrix} \arrow[ll, dotted] \arrow[ur, "{(2,1)}"'] & \\
& & & & \begin{smallmatrix}(3,5)\\5\end{smallmatrix} \arrow[ur, "{(1,1)}"'] & & & &
\end{tikzcd}$$

Here, at the position of each vertex, the name of the vertex is written in the first line, and its length in the second one.

7.3 Theorem. *Let A be a simply connected representation-finite artinian ring. Then the Auslander-Reiten quiver Γ_A of A is a translation quiver obtained in Construction 7.1. The quiver Q needed for the construction is also simply connected.*

Proof. The only thing which is not directly clear is the assertion that the arrows of Q are indeed valued with dimension sequences. To see that, we will show that $\mathrm{Hom}(M, P)$ is a dimension sequence bimodule for P an arbitrary indecomposable projective module and M any direct summand of its radical. Let P' be an indecomposable direct summand of the projective cover of M.

First, we will see that $\mathrm{Hom}(P', P)$ is a dimension sequence bimodule. Let $I = \mathrm{Tr}(A\text{-proj-ind} \setminus \{P, P'\})$ be the trace of all indecomposable projective modules which are not isomorphic to P or P', and let $B = A/I$. Then B-mod is a subcategory of A-mod and therefore B is also representation-finite. As P/IP and P'/IP' are the only projective B modules, Theorem 6.7 implies $\mathrm{Hom}(P', P)$ is a dimension sequence bimodule.

We can write B (up to Morita equivalence) as

$$\begin{pmatrix} \mathrm{End}(P') & \mathrm{Hom}(P', P) \\ 0 & \mathrm{End}(P) \end{pmatrix}$$
$$= \begin{pmatrix} \mathrm{End}(P') & \bigoplus_{\tilde{M}} \mathrm{Hom}(P', \tilde{M}) \bigotimes_{\mathrm{End}(\tilde{M})} \mathrm{Hom}(\tilde{M}, P) \\ 0 & \mathrm{End}(P) \end{pmatrix},$$

where the direct sum runs through the indecomposable direct summands of ϑP. If more than one summand in this decomposition where nonzero, then $\mathrm{Hom}(P', P)$ could not be a dimension sequence bimodule. Therefore we may assume $\mathrm{Hom}(P', P) = \mathrm{Hom}(P', M) \otimes \mathrm{Hom}(M, P)$. Now Lemma 6.11 says that $\mathrm{Hom}(M, P)$ is also a dimension sequence bimodule. $\square$

7.4 Theorem. *Let Q, d and s be as in Construction 7.1. Assume further that there are skewfields D_i for $i \in Q_0$ and dimension sequence bimodules M_{ij} for $(i \longrightarrow j) \in Q_1$ with corresponding dimension sequence $d(i \longrightarrow j)$. Then the translation quiver Γ obtained in the construction is a component of the Auslander-Reiten quiver of some artinian ring.*

Especially, if Γ is finite, then it is the Auslander-Reiten quiver of a representation-finite artinian ring.

Proof. Without loss of generality we may assume that the first τ-orbit starts last, that means $s(1) = \max_i\ s(i)$.

Inductively, there is an artinian ring B, such that the Auslander-Reiten quiver of B comes from Construction 7.1 applied to $Q \setminus \{1\}$ (with d and s restricted accordingly). Let $X_i \in B$-ind be the indecomposable B-modules in position $(i, s(1) - 1)$ in the Auslander-Reiten quiver of B-mod. Set

$$A = \begin{pmatrix} B & \oplus_i X_i \otimes_{D_i} M_{i1} \\ 0 & D_1 \end{pmatrix}.$$

If $P = \begin{pmatrix} \oplus X_i \otimes_{D_i} M_{i,1} \\ D_1 \end{pmatrix}$ it the projective A-module in the first τ-orbit, then obviously $\mathrm{Rad}(P) = \bigoplus \begin{pmatrix} X_i \\ 0 \end{pmatrix}^{dim_{D_i} M_{i,1}}$, and

$$\mathrm{Hom}_A(\begin{pmatrix} X_i \\ 0 \end{pmatrix}, P) = \mathrm{Hom}_B(X_i, \oplus X_i \otimes_{D_i} M_{i,1}) = \mathrm{Hom}_B(X_i, X_i \otimes_{D_i} M_{i,1}).$$

The map

$$\begin{aligned} {}_{D_i}M_{i1}{}_{D_1} &\longrightarrow {}_{D_i}\operatorname{Hom}(X_i, X_i \otimes_{D_i} M_{i,1})_{D_1} \\ m &\longmapsto [x \longmapsto m \otimes b\,] \end{aligned}$$

is a monomorphism of bimodules. It is surjective, since the left dimensions of both objects are identical. Hence the edges connecting P to the original quiver are exactly the ones required.

Adding up everything we find that projective modules in the Auslander-Reiten-Quiver behave just like those in the given translation quiver. Since the lengths also calculate in the same way, the quivers must be isomorphic. □

7.5 Remark. This shows that, more or less (modulo the existence of suitable skewfields and bimodules), every translation quiver which comes from a simply connected quiver and behaves correctly on a length function, is indeed the Auslander-Reiten quiver of some artinian ring. Therefore it is now possible to combinatorically draw such quivers in order to see what Auslander-Reiten quivers can look like. A computer program (see appendix B for the source code) helped to quickly do the calculation.

For a vertex v in the Auslander-Reiten quiver of an artinian ring, we will denote the corresponding indecomposable module by m_v.

7.6 Theorem. *Let Q, d, s, D_i and M_{ij} be as in Theorem 7.4. Let d^{R} be the function sendig each arrow to the dimension sequence d sends it to, but rotated cyclically one step. That means $d^{\mathrm{R}}(i \longrightarrow j)_k = d(i \longrightarrow j)_{k+1}$.*

Then $Q, d^{\mathrm{RR}}, s, D_i$ and M_{ij}^{RR} also meet the conditions of 7.4, and the unvalued quivers corresponding to the preprojective components of the different Auslander-Reiten quivers are isomorphic.

Further, if $\operatorname{Hom}(m_v, m_w) = 0$ *interpreted as homomorphisms in one ring, then* $\operatorname{Hom}(m_v, m_w) = 0$ *interpreted as homomorphisms in the other.*

Proof. It is clear that $Q, d^{\mathrm{RR}}, s, D_i$ and M_{ij}^{RR} satisfy the conditions of 7.4. Let $A, A^{\mathrm{RR}}, \Gamma$ and Γ^{RR} be the corresponding artinian rings and translation quivers.

For the first statement one only needs to find out that Γ and Γ^{RR} “stop at the same points”.

We will show, by induction on w:

$$\operatorname{Hom}_{A^{\mathrm{RR}}}(m_{(v,i)}, m_{(w,j)}) \cong \operatorname{Hom}_A(m_{(v,i)}, m_{(w,j)})^{\mathrm{RR}}$$

For $(v, i) = (w, j)$ this is true by construction of A^{RR}.

For $w = v + 1$ either $s(j) = w$ (in that case the claim is the definition) or $\mathrm{Hom}_{A^{\mathrm{RR}}}(m_{(v,i)}, m_{(w,j)}) \cong \mathrm{Hom}_{A^{\mathrm{RR}}}(m_{(w-2,j)}, m_{(v,i)})^{\mathrm{R}}$ by Theorem 3.25. Then

$$\begin{aligned}\mathrm{Hom}_A(m_{(v,i)}, m_{(w,j)})^{\mathrm{RR}} &\cong \mathrm{Hom}_A(m_{(w-2,j)}, m_{(v,i)})^{\mathrm{RRR}}\\ &\cong \mathrm{Hom}_{A^{\mathrm{RR}}}(m_{(w-2,j)}, m_{(v,i)})^{\mathrm{R}}\\ &\cong \mathrm{Hom}_{A^{\mathrm{RR}}}(m_{(v,i)}, m_{(w,j)})\end{aligned}$$

so the claim holds in that case.

Let $m_{(w-2,j)} \rightarrowtail \oplus_r m_{(w-1,r)} \longrightarrow m_{(w,j)}$ be the right almost split sequence of $m_{(w,j)}$. By the factorization property we have the following isomorphism.

$$\mathrm{Hom}_A(m_{(v,i)}, m_{(w,j)}) \cong \frac{\oplus_k \mathrm{Hom}_A(m_{(v,i)}, m_{(w-1,k)}) \otimes \mathrm{Hom}_A(m_{(w-1,k)}, m_{(w,j)})}{\oplus_r(\mathrm{Hom}_A(m_{(v,i)}, m_{(w-2,j)})\nu_r \otimes \mu_r)}$$

The map $\mathrm{Hom}_A(m_{(v,i)}, m_{(w-2,j)}) \longrightarrow \oplus_r (\mathrm{Hom}_A(m_{(v,i)}, m_{(w-2,j)})\nu_r \otimes \mu_r)$ is an isomorphism of left vector spaces and by Lemma 4.4 the right vector space structure is also the same. Therefore the denominator above is isomorphic to $\mathrm{Hom}_A(m_{(v,i)}, m_{(w-2,j)})$. By Lemma 6.10 dualizing twice commutes with tensor products. Since dualizing is exact, dualizing twice also commutes with factoring. Therefore, and because obviously everything said so far is true for A^{RR} as well, and the following equalities hold:

$$\begin{aligned}&\mathrm{Hom}_A(m_{(v,i)}, m_{(w,j)})^{\mathrm{RR}}\\ &\cong \left(\frac{\oplus_k \mathrm{Hom}_A(m_{(v,i)}, m_{(w-1,k)}) \otimes \mathrm{Hom}_A(m_{(w-1,k)}, m_{(w,j)})}{\mathrm{Hom}_A(m_{(v,i)}, m_{(w-2,j)})}\right)^{\mathrm{RR}}\\ &\cong \left(\frac{\oplus_k \mathrm{Hom}_A(m_{(v,i)}, m_{(w-1,k)})^{\mathrm{RR}} \otimes \mathrm{Hom}_A(m_{(w-1,k)}, m_{(w,j)})^{\mathrm{RR}}}{\oplus_r(\mathrm{Hom}_A(m_{(v,i)}, m_{(w-2,j)})^{\mathrm{RR}}}\right)\\ &\cong \left(\frac{\oplus_k \mathrm{Hom}_{A^{\mathrm{RR}}}(m_{(v,i)}, m_{(w-1,k)}) \otimes \mathrm{Hom}_{A^{\mathrm{RR}}}(m_{(w-1,k)}, m_{(w,j)})}{\mathrm{Hom}_{A^{\mathrm{RR}}}(m_{(v,i)}, m_{(w-2,j)})}\right)\\ &\cong \mathrm{Hom}_{A^{\mathrm{RR}}}(m_{(v,i)}, m_{(w,j)})\end{aligned}$$

Here the equality between the third and the fourth line holds inductively.

Now, if one of Γ and Γ^{RR} had a module in a position in the translation quiver where the other does not, then the above isomorphism could not hold on all positions which belong to the right almost split sequence of this one.

Note that the second assertion also follows directly from the isomorphism shown above. □

7.7 Theorem. *Let A be a simply connected representation-finite artinian ring. Then the dimension vectors of two non-isomorphic indecomposable A-modules are linearly independent.*

Proof. Assume, that the dimension vectors $[M]$ and $[N]$ of two indecomposable A-modules M and N are linearly dependent. Without loss of generality, M and N can be assumed to be sincere (otherwise compare them as modules over the ring $A/\operatorname{Tr}\{P \in A\text{-proj} : \operatorname{Hom}(P, M) = 0\}$).

A sectional path (see [11]) in a translation quiver is a path (i.e. a sequence of arrows and inverse arrows such that every arrow starts in the position where the one before stopped) without sub-paths which connect any vertex v to τv, $\tau^{-}v$ or v itself.

By induction from left to right (or over i, in the notation of Construction 7.1) one can see that the dimension vectors of the indecomposable modules on some sectional path are linearly independent. Namely, the only operations needed are adding the linearly independent dimension vector of a projective module to the set of dimension vectors one already has, and replacing the dimension vector of a module Q by the dimension vector of $\tau^{-}Q$, when all direct summands of $\vartheta^{-}Q$ are also in out set.

Therefore this is especially true for the modules on the first possible sectional path of maximal length. Since both N and M are sincere their position in the Auslander-Reiten quiver is right of this sectional path. But from this first maximal sectional path to the last one, the dimension vectors are calculated in the same way as they are in the corresponding hereditary algebra, just by starting with a different $\mathbb{Z}$-basis. Therefore it follows from Theorem 5.12 that $M \cong N$. $\square$

8 Bricks — classification of Auslander-Reiten quivers of hereditary artinian rings

Bricks where introduced by Ringel [11] to understand tubes in the Auslander-Reiten quivers of finite dimensional tame algebras over an algebraically closed field. We will use this concept to give a necessary condition for a simply connected artinian ring to be representation-finite (8.5). With this criterion, it will be possible to give a proof of the theorem of Dowbor, Ringel and Simson saying that a species is representation finite if and only if the corresponding quiver is a Coxeter diagram (8.10).

8.1 Definition. Let A be an artinian ring. Following [11], an A-module b will be called *brick* if its endomorphism ring is a skewfield. Two bricks b_1 and b_2 are called *orthogonal*, if $\mathrm{Hom}(b_1, b_2) = 0 = \mathrm{Hom}(b_2, b_1)$.

Note that in [11] the endomorphism ring of a brick is required to be the base field, but since there is no base field present in our context, the given definition is the appropriate generalization.

8.2 Theorem. *Let A be an artinian ring, $B \subset A$-mod a finite set of pairwise orthogonal bricks. Let $\mathcal{B}$ be the full subcategory of A-modules which have a filtration such that every factor of this filtration is isomorphic to a brick in B. Then $\mathcal{B}$ is a full abelian subcategory of A-mod, and the elements of B are representatives of the isomorphism classes of simple objects in $\mathcal{B}$.*

Proof. A proof can be found in [11]. Since we varied the definition of bricks slightly and since the proof is nice and simple we will nevertheless give it.

It is obvious that $\mathcal{B}$ is closed under finite direct sums. Let m and n be in $\mathcal{B}$, φ a homomorphism $m \longrightarrow n$. We will show that the kernel of φ (as morphism in A-mod) is in $\mathcal{B}$. Dually cokernels of morphism in $\mathcal{B}$ are in $\mathcal{B}$, and therefore the same is true for images.

Let $0 = m_0 < m_1 < \cdots < m_k = m$ and $0 = n_0 < n_1 < \cdots < n_l = n$ be filtrations such that $m_i/m_{i-1}, n_i/n_{i-1} \in \mathcal{B}$. We use induction on $k + l$. For $k + l = 1$ the claim is trivially true.

In the case $\varphi(m_1) = 0$ the following diagram, with exact rows and columns, commutes:

$$\begin{tikzcd}
m_1 \arrow[r, tail] \arrow[d, equal] & k \arrow[r, two heads] \arrow[d, tail] & \tilde{k} \arrow[d, tail] \\
m_1 \arrow[r, tail] \arrow[d] & m \arrow[r, two heads] \arrow[d] & \tilde{m} \arrow[d] \\
0 \arrow[r] & n \arrow[r, equal] & n
\end{tikzcd}$$

The first row is exact by the snake lemma. Inductively $\tilde{k} \in \mathcal{B}$. Therefore also $k \in \mathcal{B}$ and the claim holds.

In case $\varphi(m_1) \neq 0$ choose i minimal such that $\varphi(m_1) \subset n_i$. Then clearly the composition $m_1 \rightarrowtail n_i \twoheadrightarrow n_i/n_{i-1}$ is nonzero. Therefore $m_1 \cong n_i/n_{i-1}$ is a direct summand of n_i and n has a filtration

$$0 < \varphi(m_1) < \varphi(m_1) \oplus n_1 < \cdots < \varphi(m_1) \oplus n_{i-1} = n_i < n_{i+1} < \cdots < n$$

which also has only factors in B. Therefore we may assume $\varphi(m_1) = n_1$. Then the following diagram, with exact rows and columns, commutes:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, tail] & k \arrow[r, equal] \arrow[d, tail] & k \arrow[d, tail] \\
m_1 \arrow[r, tail] \arrow[d, equal] & m \arrow[r, two heads] \arrow[d] & \tilde{m} \arrow[d] \\
n_1 \arrow[r, tail] & n \arrow[r, two heads] & \tilde{n}
\end{tikzcd}$$

Since the kernel k of $m \longrightarrow n$ is also the kernel of the morphism $\tilde{m} \longrightarrow \tilde{n}$ it is inductively in $\mathcal{B}$. $\square$

8.3 Theorem. *Let A be a representation-finite artinian ring, $\mathcal{B}$ a full abelian subcategory of A-mod. Then $\mathcal{B}$ is a strict τ-category.*

Proof. One can almost copy the proof of Theorem 3.20. It is only necessary to restrict the various objects occurring to objects in $\mathcal{B}$. Especially, one should note that an object p is projective in $\mathcal{B}$, if $\operatorname{Ext}^1_A(p, b) = 0$ for any brick $b \in B$. $\square$

8.4 Theorem. *Let A be a representation-finite artinian ring, $B = \{b_1, b_2\}$ a set of two orthogonal bricks with $\operatorname{Ext}^1(b_1, b_1)$, $\operatorname{Ext}^1(b_1, b_2)$, and $\operatorname{Ext}^1(b_2, b_2)$ are all zero, $\mathcal{B}$ as in Theorem 8.2.*

Let R be the hereditary artinian ring $\begin{pmatrix} D(b_1) & Ext^1(b_2, b_1)^{\mathrm{R}} \\ 0 & D(b_2) \end{pmatrix}$.

Then the Auslander-Reiten quiver of $\mathcal{B}$ and the Auslander-Reiten quiver of R-mod are isomorphic as valued translation quivers.

Proof. Clearly the simple R-modules are

$$s_1 = \begin{pmatrix} D(b_1) \\ 0 \end{pmatrix} \text{ and } s_2 = \begin{pmatrix} 0 \\ D(b_2) \end{pmatrix}$$

and the endomorphism skewfield of each s_i is isomorphic to $D(b_i)$. The objects b_1 and s_1 are both projective in their categories. The other projective R-module is

$$p = \begin{pmatrix} Ext^1(b_2, b_1)^{\mathrm{R}} \\ D(b_2) \end{pmatrix}.$$

For the construction of the corresponding object in $\mathcal{B}$, let $\mathbb{E}_1, \dots, \mathbb{E}_k$ be a $D(b_1)$ basis of $\mathrm{Ext}^1(b_2, b_1)$, and regard $\oplus\mathbb{E}_i :\ b_1^{(k)} \rightarrowtail q \twoheadrightarrow b_2$. We claim that q is a projective indecomposable object in $\mathcal{B}$. If that is true q especially is the projective cover of b_2. We have the exact sequences

$$\underbrace{\mathrm{Ext}^1(b_2, b_2)}_{=0} \longrightarrow \mathrm{Ext}^1(q, b_2) \longrightarrow \underbrace{\mathrm{Ext}^1(b_1^{(k)}, b_2)}_{=0}$$

and

$$\underbrace{\mathrm{Hom}(b_2, b_1)}_{=0} \rightarrowtail \mathrm{Hom}(q, b_1) \longrightarrow \mathrm{Hom}(b_1^{(k)}, b_1) \xrightarrow{\xi} \mathrm{Ext}^1(b_2, b_1)$$
$$\longrightarrow \mathrm{Ext}^1(q, b_1) \longrightarrow \underbrace{\mathrm{Ext}^1(b_1^{(k)}, b_1)}_{=0}.$$

Since ξ is an isomorphism by construction, it follows that $\mathrm{Ext}^1(q, b_2) = 0$, $\mathrm{Hom}(q, b_1) = 0$ and $\mathrm{Ext}^1(q, b_1) = 0$. Therefore q is projective. If q were decomposable, then it would have a direct summand isomorphic to b_1, contradicting $\mathrm{Hom}(q, b_1) = 0$.

Endomorphisms of b_2 lift uniquely to ones of q, since the difference between any two possible lifts would have to factor through $b_1^{(k)}$, therefore $\mathrm{End}(q) \cong D(b_2) \cong \mathrm{End}_R(p)$ as skewfields. This also induces an action of endomorphisms of b_2 on $b_1^{(k)}$ by taking the kernel map. With this action, ${}_A(b_1^{(k)})_{D(b_2)} = b_1 \otimes \mathrm{Hom}(b_1, q)$ as bimodules. Also $\mathrm{Ext}(b_2, b_1) = \mathrm{Hom}(b_1^{(k)}, b_1)$ as bimodules, because $b_1^{(k)}$ is the syzygy of b_2 in $\mathcal{B}$ and there are no morphisms factoring through q, which one would need to factor out. Therefore

$$\begin{aligned}\mathrm{Ext}^1(b_2, b_1) &= \mathrm{Hom}(b_1 \otimes \mathrm{Hom}(b_1, q), b_1) = \mathrm{Hom}(\mathrm{Hom}(b_1, q), \mathrm{Hom}(b_1, b_1)) \\ &= \mathrm{Hom}(b_1, q)^{\mathrm{L}}\end{aligned}$$

Now $\mathrm{Hom}(b_1, q) \cong \mathrm{Hom}(s_1, p)$ as bimodules. Therefore the components of the projective objects in the two Auslander-Reiten quivers are isomorphic. Since there are only finitely many isomorphism classes of indecomposable objects in $\mathcal{B}$ these components are finite, so they are the entire quivers. □

8.5 Corollary. *Let A be a representation-directed representation-finite artinian ring, $m, n \in A$-ind orthogonal bricks. Then $\mathrm{Ext}^1_A(m, n)$ is a dimension sequence bimodule.*

Proof. We may assume $\mathrm{Ext}^1(m,n) \neq 0$ and therefore $\mathrm{Ext}^1(n,m) = 0$. Then, by Theorem 8.4, the ring $\begin{pmatrix} D(n) & Ext^1(m,n)^{\mathrm{R}} \\ 0 & D(m) \end{pmatrix}$ is representation-finite, and by Theorem 6.7 it follows that $Ext^1(m,n)$ is a dimension sequence bimodule. □

8.6 Lemma. *Let $\mathcal{S}$ be a connected species, such that at least two of the bimodules are not $(1,1)$-dimensional. Then $\mathcal{S}$ is not representation-finite.*

Proof. By applying the Coxeter functors (5.6) if necessary, $\mathcal{S}$ can be assumed to have a subspecies of the form $D_1 \xrightarrow{M_{1,2}} D_2 \xrightarrow{M_{2,3}} \cdots \xrightarrow{M_{n-1,n}} D_n$, with $M_{1,2}$ and $M_{n-1,n}$ not $(1,1)$ dimensional, which will be called $\mathcal{S}'$. If one of $M_{1,2}$ and $M_{n-1,n}$ is no dimension sequence bimodule the species is clearly representation infinite. Therefore we may assume d and e to be the corresponding dimension sequences, $d \neq (1,1,1) \neq e$.

We have the following two indecomposable $\mathcal{S}'$-representations, which are clearly orthogonal.

$$\begin{aligned} b_1 &= D_1 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0, \\ b_2 &= 0 \longrightarrow M_{23} \otimes \cdots \otimes M_{n-1,n} \longrightarrow \cdots \longrightarrow M_{n-1,n}. \end{aligned}$$

Since $q = M_{12} \otimes \cdots \otimes M_{n-1,n} \longrightarrow M_{23} \otimes \cdots \otimes M_{n-1,n} \longrightarrow \cdots \longrightarrow M_{n-1,n}$ is the projective cover of b_2 we have $\mathrm{Ext}^1(b_2,b_1) \cong (M_{12} \otimes \cdots \otimes M_{n-1,n})^{\mathrm{L}}$ (see the proof of 8.4). Then by Lemma 6.10

$$\begin{aligned} \mathrm{Ext}^1(b_2,b_1)^{i\mathrm{R}} &= (M_{12}^{(i-1)\mathrm{R}} \otimes \cdots \otimes M_{n-1,n}^{(i-1)\mathrm{R}}) \text{ for } i \text{ odd, and} \\ \mathrm{Ext}^1(b_2,b_1)^{i\mathrm{R}} &= (M_{n-1,n}^{(i-1)\mathrm{R}} \otimes \cdots \otimes M_{12}^{(i-1)\mathrm{R}}) \text{ for } i \text{ even.} \end{aligned}$$

Assume that $\mathcal{S}$ is representation-finite. Then so is $\mathcal{S}'$, and, by Corollary 8.5, $\mathrm{Ext}^1(b_2,b_1)$ is a dimension sequence bimodule. Let f be the corresponding dimension sequence. Then $f_i \geq d_i e_i \ \forall i$. Therefore the neighbors of any "1" in the sequence f are at least 4. Applying the inverse operation of the construction of the dimension sequences in Definition 6.1, one finds a dimension sequence with all entries larger than 1, a contradiction. □

8.7 Lemma. *For the species $\mathcal{S} : D \xrightarrow{D} \cdots \xrightarrow{D} D \xrightarrow{M} E \xrightarrow{E} \cdots \xrightarrow{E} E$ let $\mathcal{S}^{\frac{1}{2}\mathrm{R}}$ be the species $E \xrightarrow{E} \cdots \xrightarrow{E} E \xrightarrow{M^{\mathrm{R}}} D \xrightarrow{D} \cdots \xrightarrow{D} D$ (this is supposed to symbolize, that the number of "D"s in $\mathcal{S}$ is equal to the number of "E"s in $\mathcal{S}^{\frac{1}{2}\mathrm{R}}$ and vice versa). Then $\mathcal{S}$ and $\mathcal{S}^{\frac{1}{2}\mathrm{R}}$ have isomorphic unvalued preprojective components of Auslander-Reiten quivers. Further*

$$\mathrm{Hom}_{\mathcal{S}}(m_v,m_w)^{\mathrm{R}} \cong \mathrm{Hom}_{\mathcal{S}^{\frac{1}{2}\mathrm{R}}}(m_v,m_w)$$

for preprojective vertices v and w in the Auslander-Reiten quiver.

Proof. Copy the proof of Theorem 7.6. It is worth noting, that the species must have this favorite shape in order to have a canonical candidate of what to call $\mathcal{S}^{\frac{1}{2}\mathrm{R}}$. $\square$

We will need the following technical lemma. It tells us that a certain construction does not yield dimension sequences to often. This can be applied in order to show that some species cannot be representation-finite in the proof of the following theorem.

8.8 Lemma. *Let d be a dimension sequence such that $(d_n d_{n+1} - 1)_n$ is equivalent to a dimension sequence. Then the length of d is at most 5.*

Proof. By Remark 6.3, any dimension sequence can successively be reduced to $(0,0)$, by eliminating "1"s and decreasing their neighbors. Using this, one can see, that every nonzero dimension sequence has a subsequence of one of the forms $(1,1)$, $(1,2,1)$, $(1,2,2)$, $(1,2,3)$, $(1,3,1)$, $(2,1,2)$, $(2,1,3)$, $(2,1,4)$ or $(1,5,1)$ or one of them in the inverse order.

Let c be the dimension sequence equivalent to $(d_n d_{n+1} - 1)_n$. If $c = (0,0)$ then $d = (1,1,1)$. If c contains the subsequence $(1,1)$ then $c = (1,1,1)$ and therefore $d \in \{(1,2,1,2),(2,1,2,1)\}$.

If c contains a subsequence $(1,2,a), a \in \{1\dots 3\}$, then d contains a subsequence $(2,1,3,\frac{a+1}{3})$, and therefore $a = 2$. If the length of d is greater than 5, then d contains a subsequence $(d_i, 2, 1, 3, 1, d_{i+5})$. Applying the inverse construction of the definition of dimension sequences one finds a dimension sequence with a subsequence $(b_i, 1, 1, b_{i+5} - 1)$, but there is no such dimension sequence.

If c contains a subsequence $(1,3,1)$, then d has to contain a subsequence $(1,2,2,1)$. Reducing twice one gets a dimension sequence containing $(1,1)$. This can only be $(1,1,1)$, therefore d has length 5.

If c contains a subsequence $(2,1,a), a \in \{2\dots 4\}$, then d contains a subsequence $(3,1,2,\frac{a+1}{2})$ and therefore $a = 3$. Reducing twice one gets, again, a dimension sequence which contains $(1,1)$.

Finally, if c contains $(1,5,1)$ then there is no candidate for d at all. $\square$

8.9 Theorem. *The species $D \xrightarrow{D} D \xrightarrow{M} E$ is representation-finite if and only if M is a dimension sequence bimodule such that the corresponding dimension sequence has length at most five.*

Proof. Let $\mathcal{S}$ be the species we are talking about, and let $\mathcal{S}'$ be the subspecies $D \xrightarrow{M} E$. Clearly M needs to be a dimension sequence bimodule, let d be the corresponding dimension sequence. Let i be the injective $\mathcal{S}'$-representation $D \longrightarrow M^{\mathrm{L}}$, and let $b_2 = \tau_{\mathcal{S}'}(i)$ and $b_1 = s_1 = D \longrightarrow 0 \longrightarrow 0$. Then b_1 and b_2 are orthogonal bricks.

Let $\begin{pmatrix}0\\a\\b\end{pmatrix}$ be the dimension vector of d_2, i.e. we have an exact sequence $s_2^{(a)} \rightarrowtail b_2 \twoheadrightarrow s_3^{(b)}$ with $s_2 = 0 \longrightarrow D \longrightarrow 0$, $s_3 = 0 \longrightarrow 0 \longrightarrow E$. Then, as $\mathrm{Ext}^1(s_3, b_1) = 0$ and $\mathrm{Ext}^2 = 0$, we have

$$\dim_{D(b_1)} \mathrm{Ext}^1(b_2, b_1) = \dim_{D(b_1)} \mathrm{Ext}^1(s_2^{(a)}, b_1) = a.$$

By Theorem 4.5 $\mathrm{Ext}^1(b_2, b_1)^{k\mathrm{R}} \cong \mathrm{Hom}(b_1, \tau b_2)^{(k-1)\mathrm{R}}$.

Let p_i be the position of b_i in the Auslander-Reiten quiver of $\mathcal{S}$, $\hat{p}_2$ the position of b_2 in the Auslander-Reiten quiver of $\mathcal{S}'$. Then $m(\mathcal{S}^{\frac{k}{2}\mathrm{R}})_{p_2} = m((\mathcal{S}')^{\frac{k}{2}\mathrm{R}})_{\hat{p}_2}$, since by Lemma 8.7 the vertices in the Auslander-Reiten quiver of $\mathcal{S}$ corresponding to representations of $\mathcal{S}'$ are the same as the ones in the Auslander-Reiten quiver of $\mathcal{S}^{\frac{k}{2}\mathrm{R}}$ corresponding to representations of $(\mathcal{S}')^{\frac{k}{2}\mathrm{R}}$. Therefore

$$\begin{aligned}
\mathrm{Ext}^1_{\mathcal{S}}(b_2, b_1)^{k\mathrm{R}} &\cong \mathrm{Hom}_{\mathcal{S}}(b_1, \tau b_2)^{(k+1)\mathrm{R}} \cong \mathrm{Hom}_{\mathcal{S}^{\frac{k}{2}\mathrm{R}}}(m_{p_1}, m_{\tau p_2})^{\mathrm{R}} \\
&\cong \mathrm{Ext}^1_{\mathcal{S}^{\frac{k}{2}\mathrm{R}}}(m_{p_2}, m_{p_1}) = \mathrm{Ext}^1_{\mathcal{S}^{\frac{k}{2}\mathrm{R}}}(m((\mathcal{S}')^{\frac{k}{2}\mathrm{R}})_{\hat{p}_2}, m_{p_1}),
\end{aligned}$$

and, by the above argument, $\dim_{\mathrm{left}} \mathrm{Ext}^1_{\mathcal{S}}(b_2, b_1)^{k\mathrm{R}} = c_k$, if the dimension vector of $m((\mathcal{S}')^{k\mathrm{R}})_{\hat{p}_2}$ is $\begin{pmatrix}0\\c_k\\\star\end{pmatrix}$. The right end of the Auslander-Reiten quiver of $\mathcal{S}'$ is

$$\begin{tikzcd}
 & b_2 \arrow[dr, "{(d_{-3},d_{-2})}"] & & i \arrow[ll, dotted] \arrow[dr, "{(d_{-1},d_0)}"] & \\
\cdots & & \tau s_3 \arrow[ur, "{(d_{-2},d_{-1})}"] & & s_3 \arrow[ll, dotted]
\end{tikzcd}$$

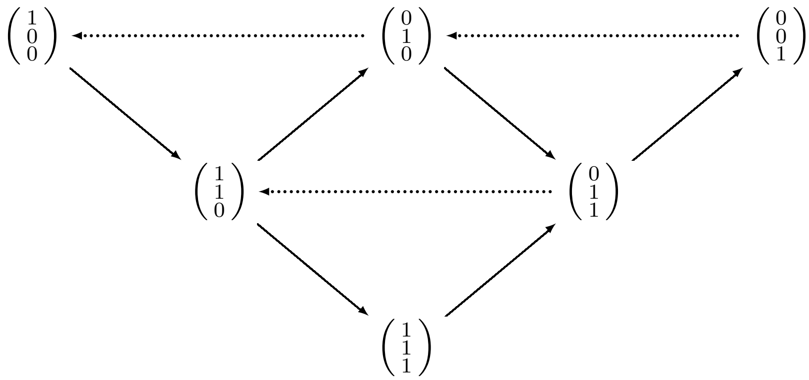
with dimension vectors

$$\begin{tikzcd}
 & \begin{pmatrix}0\\d_{-1}d_{-2}-1\\\star\end{pmatrix} \arrow[dr, "{(d_{-3},d_{-2})}"] & & \begin{pmatrix}0\\1\\d_0\end{pmatrix} \arrow[ll, dotted] \arrow[dr, "{(d_{-1},d_0)}"] & \\
\cdots & & \begin{pmatrix}0\\d_{-1}\\\star\end{pmatrix} \arrow[ur, "{(d_{-2},d_{-1})}"] & & \begin{pmatrix}0\\0\\1\end{pmatrix} \arrow[ll, dotted]
\end{tikzcd}$$

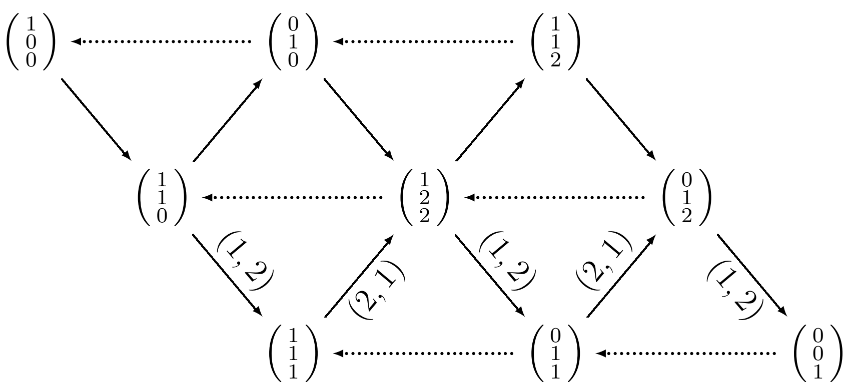
The analog with permuted indices holds for $(\mathcal{S}')^{\frac{k}{2}\mathrm{R}}$ instead of $\mathcal{S}'$. Therefore $c_k = d_{k-1}d_{k-2} - 1$. Now, if $\mathcal{S}$ is representation-finite, then (c_k) is equivalent to a dimension sequence. By Lemma 8.8, this is only possible if (d_k) is a dimension sequence of length at most five.

The converse can be verified by drawing the Auslander-Reiten quivers for $d \in \{(1,1,1), (1,2,1,2), (1,2,2,1,3)\}$. Note that it is not necessary to try the different cyclic permutations by Lemma 8.7. In the following pictures the irreducible modules are represented by their dimension vectors.

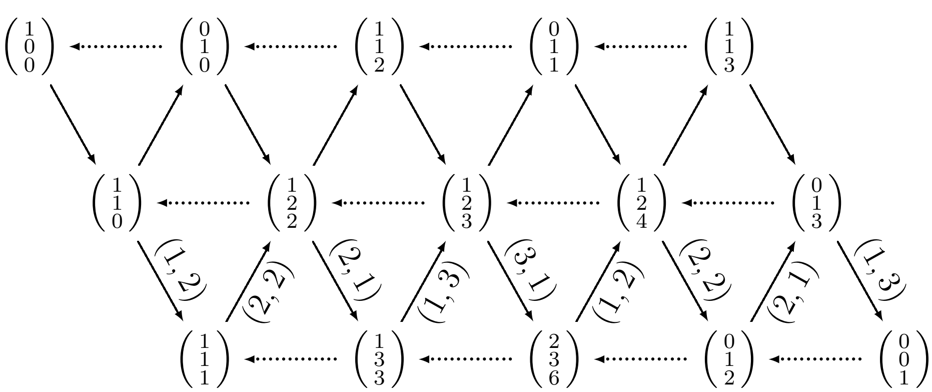
– $d = (1, 1, 1)$:



– $d = (1, 2, 1, 2)$:



– $d = (1, 2, 2, 1, 3)$:



This completes the proof. □

8.10 Theorem. *A species is representation-finite if and only if its corresponding quiver, where an edge is labeled by the length of the dimension sequence of the bimodule belonging to the edge, is a Coxeter diagram (see the list in Appendix A).*

Proof. By Theorems 8.6, 8.7 and 8.9 it is sufficient to show the following points:

1. Any species with bimodules of type $\circ \longrightarrow \circ \xrightarrow{(1,2,1,2)} \circ \longrightarrow \circ \longrightarrow \circ$ is representation infinite.

2. All species of types $\circ \xrightarrow{(1,2,1,2)} \circ \longrightarrow \cdots \longrightarrow \circ$ are representation-finite.

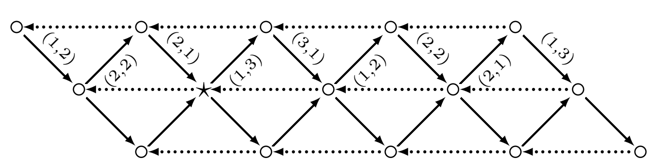
3. Species of one of the types $\circ \longrightarrow \circ \xrightarrow{(1,2,1,2)} \circ \longrightarrow \circ$, $\circ \xrightarrow{(1,2,2,1,3)} \circ \longrightarrow \circ$ or $\circ \xrightarrow{(1,2,2,1,3)} \circ \longrightarrow \circ \longrightarrow \circ$ are representation-finite.

4. All species of one of the types $\circ \xrightarrow{(1,2,2,1,3)} \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ$ or $\circ \longrightarrow \circ \xrightarrow{(1,2,2,1,3)} \circ \longrightarrow \circ$ are representation infinite.

The species containing only dimension sequences $(1,1,1)$ and $(1,2,1,2)$ occur in the representation theory of finite dimensional algebras over a field. By [2], they are representation-finite if and only if the quiver weighted with the length of the dimension sequences is a Dynkin diagram. (This will also follow from the proof of Theorem 9.10.)

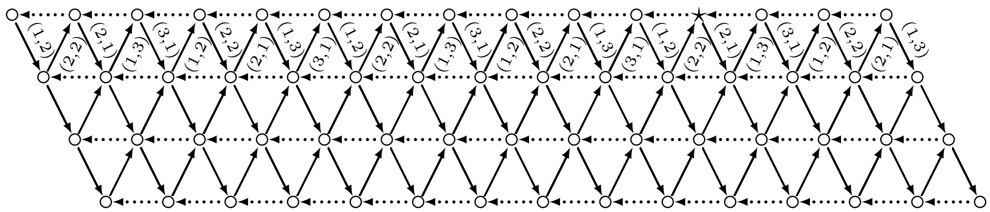
The representation-finiteness of species of types $\circ \xrightarrow{(1,2,2,1,3)} \circ \longrightarrow \circ$ and $\circ \xrightarrow{(1,2,2,1,3)} \circ \longrightarrow \circ \longrightarrow \circ$ can be verified by calculating the Auslander-Reiten quivers (see appendix C).

To see that a species of type $\circ \longrightarrow \circ \xrightarrow{(1,2,2,1,3)} \circ \longrightarrow \circ$ is representation infinite consider the orthogonal bricks $b_1 = s_1 = D_1 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$ (where D_1 is the appropriate skewfield) and b_2 the $\circ \xrightarrow{(1,2,2,1,3)} \circ \longrightarrow \circ$-representation in the position marked $\star$ in the Auslander-Reiten quiver.



Then by the argument in the proof of 8.9 the sequence $(\dim_{\text{left}} \operatorname{Ext}^1(b_2, b_1)^{k\mathrm{R}})_k$ can be read off from the dimension vectors in Appendix C. It is equivalent to $(2,3,3,2,4)$, and therefore $\operatorname{Ext}^1(b_2, b_1)$ is not a dimension sequence bimodule.

For a species of type $\circ \xrightarrow{(1,2,2,1,3)} \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ$ let b_1 be the $\circ \xrightarrow{(1,2,2,1,3)} \circ \longrightarrow \circ \longrightarrow \circ$-representation in the marked position and $b_2 = s_5 = 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow D_5$.



Then $(\dim_{\mathrm{left}} \operatorname{Ext}(b_2, b_1)^{k\mathrm{R}})_k \sim (2, 6, 1, 6, 2)$ (again by Appendix C), and therefore $\operatorname{Ext}(b_2, b_1)$ is no dimension sequence bimodule. $\square$

8.11 Remark. The preceding theorem is stated in [5], but the authors only give a rather rough sketch of a proof and remark, that their proof is "rather technical". Especially, they do not say how they showed the representation infinite species to be so.

9 The root system corresponding to the Coxeter diagram

In the last section it was shown that the representation-finite species are exactly the ones corresponding to Coxeter diagrams. For finite dimensional algebras over a field the situation is even better: The indecomposable representations are in bijection to the positive roots of the Coxeter diagram. It will be show in 9.10 that this still holds for arbitrary representation-finite species. Unfortunately, the proof is not direct but uses the classifications.

First note that the construction in 7.1 does not depend on the $d(i \longrightarrow j)$ to be dimension sequences. It is sufficient to have arbitrary sequences in a linearly ordered ring. Here constant real sequences will be considered.

Construction 7.1 gives the Auslander-Reiten quiver of a species if $s(j) - s(i) = 1$ whenever there is an arrow $i \longrightarrow j$. Therefore we will assume s to have that property and therefore not specify it in this section.

9.1 Example. $\lambda = \sqrt{2}$. Then $\circ \xrightarrow{\lambda} \circ$ gives the translation quiver

$$\begin{tikzcd}
1 \arrow[dr, "{(\lambda,\lambda)}"] & & 1+\lambda \arrow[ll, dotted] \arrow[dr, "{(\lambda,\lambda)}"] & \\
& 1+\lambda \arrow[ur, "{(\lambda,\lambda)}"] & & 1 \arrow[ll, dotted]
\end{tikzcd}$$

In a translation quiver like this, the arrows will sometimes only be labeled λ instead of (λ, λ) for clarity.

For a real valued quiver Q, the translation quiver arising in this construction will be called $T(Q)$. For a quiver Q with edges labeled natural numbers ≥ 2 we will call $T(Q)$ the translation quiver which is obtained by first going to the quiver valued $2\cos\frac{\pi}{\text{old value}}$ and then doing the above construction. By this convention the translation quiver in Example 9.1 is called $T(\circ \xrightarrow{\lambda} \circ)$ as well as $T(\circ \xrightarrow{4} \circ)$.

We will also need to consider the (always infinite) translation quiver $\widetilde{T}(Q)$ which is constructed like $T(Q)$, but without the "cutting off of vertices with negative length". In the context of Example 9.1 that would be

$$\widetilde{T}(Q): \quad \begin{tikzcd}
1 \arrow[dr, "{(\lambda,\lambda)}"] & & 1+\lambda \arrow[ll, dotted] \arrow[dr, "{(\lambda,\lambda)}"] & & -1 \arrow[ll, dotted] \arrow[dr, "{(\lambda,\lambda)}"] & & -1-\lambda \arrow[ll, dotted] \arrow[dr, "{(\lambda,\lambda)}"] & & 1 \arrow[ll, dotted] & \cdots \\
& 1+\lambda \arrow[ur, "{(\lambda,\lambda)}"] & & 1 \arrow[ll, dotted] \arrow[ur, "{(\lambda,\lambda)}"] & & -1-\lambda \arrow[ll, dotted] \arrow[ur, "{(\lambda,\lambda)}"] & & -1 \arrow[ll, dotted] \arrow[ur, "{(\lambda,\lambda)}"] & &
\end{tikzcd}$$

We want to consider the quivers $T(Q)$ and $\widetilde{T}(Q)$ with vertices labeled an analogon of dimension vectors instead of an analogon of lengths as well. For the quiver from 9.1 that would be

$$\begin{tikzcd}
\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \arrow[dr, "{(\lambda,\lambda)}"] & & \left(\begin{smallmatrix}1\\\lambda\end{smallmatrix}\right) \arrow[ll, dotted] \arrow[dr, "{(\lambda,\lambda)}"] & \\
& \left(\begin{smallmatrix}\lambda\\1\end{smallmatrix}\right) \arrow[ur, "{(\lambda,\lambda)}"] & & \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \arrow[ll, dotted]
\end{tikzcd}$$

We will call these quivers $T(Q)$ and $\widetilde{T}(Q)$ as well and refer to the different possible labels as the length or the dimension vector of a vertex.

9.2 Definition. For a complete quiver with n vertices Q with edges valued natural numbers $m_{ij} \geq 2$ (2 can and will be represented by "no arrow", especially if we talk about properties like "connected") the *Coxeter group G* is defined as $\langle s_1, \ldots, s_n \mid s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle$.

9.3 Theorem. *Let $V \cong \mathbb{R}^n$ with basis $\{e_i\}$ and bilinear form $(-,-)$ defined by $(e_i, e_i) = 1$ and $(e_i, e_j) = -\cos \frac{\pi}{m_{ij}}$ for $i \neq j$. Then $G \longrightarrow \mathrm{GL}(V)$: $s_i \longmapsto \sigma_i$ is a monomorphism of groups, where σ_i is the orthogonal reflection with respect to the hyper-plane orthogonal to e_i.*

Proof. See [4], Chapter V, §4 or [8], Corollary 5.4. $\square$

9.4 Theorem. *The Coxeter group G is finite if and only if the corresponding diagram is the disjoint union of Coxeter diagrams. (See Appendix A for a list of the Coxeter diagrams).*

Proof. See [4] or [8]. $\square$

9.5 Definition. If G is a finite Coxeter group then

$$\mathcal{R} = \{ r \in V \setminus \{0\} \mid x \longmapsto x - 2(x,r)r \text{ is in } G \}$$

is called the *root system of G*.

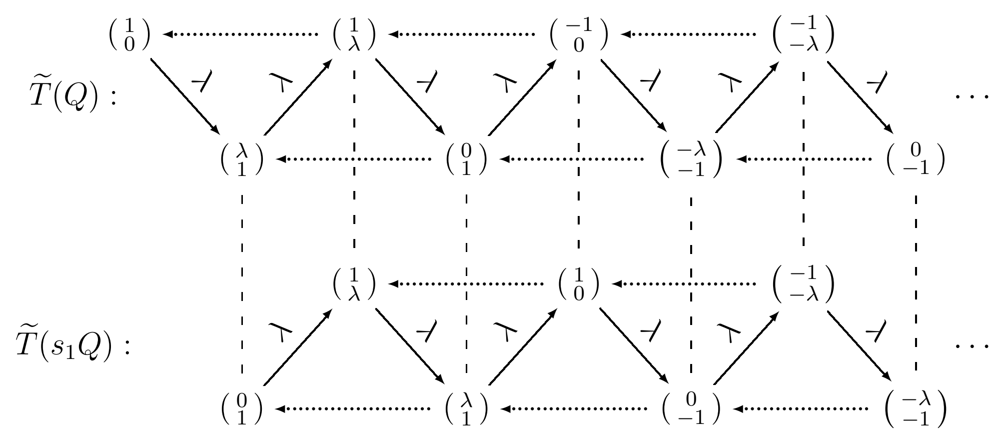
A root $r \in \mathcal{R}$ is called *positive* if it is positive as a vector, i.e. if it is nonzero and has no negative entries.

This is consistent with the definition of root systems in [8], in [4] root systems are always required to have the property $(r,s) \in \frac{1}{2}\mathbb{Z}\ \forall r,s \in \mathcal{R}$. However, most of what is said in [4] about root systems remains true.

Next we want to formally apply something similar to Construction 5.6 to these new translation quivers. Obviously it does not make sense to search for functors, as there are no categories present, but just for isomorphisms between large parts of the different translation quivers.

9.6 Construction. Let Q be a quiver (with edges labeled real or natural numbers) and i a source of Q. Then let s_iQ be the quiver obtained from Q by reversing all edges $i \longrightarrow j$. The vertices of $\widetilde{T}(Q)$ other than $(i, s(i))$ can canonically be identified with those of $\widetilde{T}(s_iQ)$.

The formal definition of this identification is to sent a vertex (i, n) in the translation quiver $\widetilde{T}(Q)$ to (i, n) in the translation quiver $\widetilde{T}(s_iQ)$. But the idea probably gets clearer by looking at an example. For Q as in Example 9.1 this identification is as symbolized by the dashed lines below.



One could also look at the identifications in the single steps of Example 5.10, if one just looks at the translation quivers and forgets that a species is present.

The inverse construction can be applied if i is a sink, and the obtained quiver will also be denoted by s_iQ. If i meets one of the conditions clearly $s_i^2Q = Q$.

9.7 Lemma. *Let Q be a simply connected quiver with edges valued natural numbers ≥ 3. The identification of vertices of $\widetilde{T}(Q)$ with vertices of $\widetilde{T}(s_iQ)$ induces the map σ_i (i.e. orthogonal reflection with respect to the hyper-plane orthogonal to e_i) on dimension vectors.*

Proof. Without loss of generality the vertex i can be assumed to be a source. Let j be any vertex of Q.

First assume there is an edge $i \xrightarrow{m_{ij}} j$. Let $\lambda = \cos \frac{\pi}{m_{ij}}$. Then the dimension vector d at position $(j, s(j))$ has i-th coordinate 2λ and j-th coordinate 1. Therefore the $(d, e_i) = 2\lambda + (e_j, e_i) = \lambda$ and the i-th coordinate of $\sigma_i b$ is 0. Clearly the other coordinates are neither affected by the identification in 9.6, nor by the reflection with respect to e_i.

For $j = i$ we find that the vector in position $(i, s(i) + 1)$ is $\sum d_k - e_i$ and therefore correctly mapped to $\sum \sigma_i d_k + e_i$.

If j is not i and no neighbor of i the dimension vector in the position $(j, s(j))$ is the sum of scalar multiples of the vectors at the positions $(k, s(k))$ for k a neighbors of i and a vector with 0 in the i-th and all k-th coordinates. Therefore these vectors are also transformed correctly.

The remaining dimension vectors are linear combinations of dimension vectors, for which the claim inductively holds. $\square$

9.8 Corollary. *Assume* $Q = \circ \xrightarrow{m_{1,2}} \cdots \xrightarrow{m_{n-1,n}} \circ$. *Then the vertex* (i, k) *in* $\widetilde{T}(Q)$ *has dimension vector* $(\sigma_1 \cdots \sigma_n)^{k-s(i)} \sigma_1 \cdots \sigma_{i-1} e_i$.

9.9 Corollary. *If* Q *is a Coxeter diagram, then the dimension vectors of the vertices of* $T(Q)$ *are exactly the positive roots. Otherwise* $T(Q) = \widetilde{T}(Q)$.

9.10 Theorem. *Let* $\mathcal{S}$ *be a species. Then* $\mathcal{S}$ *is representation-finite if and only if the translation quiver coming from the quiver underlying* $\mathcal{S}$ *with edges valued length*(d) *(with* d *the dimension sequence corresponding to the module belonging to this edge) is. In that case the two unvalued translation quivers are the same. In particular this induces a bijection between the isomorphism classes of indecomposable representations and the positive roots in the root system corresponding to this underlying quiver.*

Proof. First note that this construction does not affect edges valued $(0, 0)$ or $(1, 1, 1)$. Therefore, the assertion is true for all quivers which do not have an edge with a dimension sequence of length ≥ 4. It is also true, if the quiver Q does not represent a Coxeter diagram, as we know that then both translation quivers are infinite.

Now let us assume that the dimension sequence of length ≥ 4 is equivalent to $(1, 2)$. Then, without loss of generality,

$$\mathcal{S} \mathrel{\hat{=}} \circ_1 \longrightarrow \cdots \longrightarrow \circ_n \xrightarrow{(1,2,1,2)} \circ_{n+1} \longrightarrow \cdots \longrightarrow \circ_{n+m}, \text{ and}$$
$$\mathcal{S}^{\frac{1}{2}\mathrm{R}} \mathrel{\hat{=}} \circ_1 \longrightarrow \cdots \longrightarrow \circ_n \xrightarrow{(2,1,2,1)} \circ_{n+1} \longrightarrow \cdots \longrightarrow \circ_{n+m}.$$

Now the values of the length function l for the translation quiver coming from $\circ_1 \longrightarrow \cdots \longrightarrow \circ_n \xrightarrow{\sqrt{2}} \circ_{n+1} \longrightarrow \cdots \longrightarrow \circ_{n+m}$ can be transformed to the function l_1 corresponding to $\mathcal{S}$ and l_2 corresponding to $\mathcal{S}^{\frac{1}{2}\mathrm{R}}$ by

$$\Psi_1 : a\sqrt{2} + b \longmapsto \begin{cases} 2a + b & \text{vertex above the critical edge} \\ a + b & \text{vertex below the critical edge} \end{cases}, \text{ and}$$

$$\Psi_2 : a\sqrt{2} + b \longmapsto \begin{cases} a + b & \text{vertex above the critical edge} \\ 2a + b & \text{vertex below the critical edge} \end{cases}, \text{ respectively.}$$

See Table 2 on page 55 as an illustration of these functions. This is true for the starting points $(i, s(i))$ and can easily be seen by induction for the rest of the quiver.

Since the unvalued translation quivers corresponding to $\mathcal{S}$ and $\mathcal{S}^{\frac{1}{2}\mathrm{R}}$ are the same by Lemma 8.7, and the implications $2a+b>0 \wedge a+b>0 \Rightarrow a\sqrt{2}+b>0$ and $a\sqrt{2}+b>0 \Rightarrow 2a+b>0 \vee a+b>0$ clearly hold, the unvalued translation quiver coming from $\circ_1 \longrightarrow \cdots \longrightarrow \circ_n \xrightarrow{\sqrt{2}} \circ_{n+1} \longrightarrow \cdots \longrightarrow \circ_{n+m}$ is also identical to them.

Next we assume $\mathcal{S}$ to have exactly two vertices. Then $\mathcal{S} \triangleq \circ \xrightarrow{d} \circ$ for a dimension sequence d. Let $l = \operatorname{length}(d), \lambda = 2\cos\frac{\pi}{l}$. Let $p_n(x)$ be the n-th polynomial in the following "generic quiver".

$$\begin{tikzcd}
1 \arrow[dr, "x"] & & x^2+x-1 \arrow[ll, dotted] \arrow[dr, "x"] & & \cdots \\
& x+1 \arrow[ur, "x"] & & x^3+x^2-2x-1 \arrow[ll, dotted] &
\end{tikzcd}$$

That means $p_1(x) = 1$, $p_2(x) = x+1$ and $p_n(x) = xp_{n-1}(x) - p_{n-2}(x) \forall n \geq 3$. Then the claim is $p_{l+1}(\lambda) \leq 0$ and $p_n(\lambda) > 0$ for $1 \leq n \leq l$.

For $q_0(x) = 0$, $q_1(x) = 1$ and $q_n(x) = xq_{n-1}(x) - q_{n-2}(x) \forall n \geq 2$ one can see, by induction, that $p_n = q_n + q_{n-1}$. The recursion formula can be reformulated as

$$\begin{pmatrix} q_n(x) \\ q_{n+1}(x) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} \begin{pmatrix} q_{n-1}(x) \\ q_n(x) \end{pmatrix}.$$

For $\mu_{k,n} = 2\cos\frac{k\pi}{n}, 1 \leq k < n$ the characteristic polynomial of $\left(\begin{smallmatrix} 0 & 1 \\ -1 & \mu_{k,n} \end{smallmatrix}\right)$ is $x(x-\mu_{k,n}) + 1 = x^2 - 2\cos\frac{k\pi}{n}x + 1 = (x - e^{\frac{k\pi}{n}i})(x - e^{-\frac{k\pi}{n}i})$. Therefore the eigenvalues of this matrix are $e^{\frac{k\pi}{n}i}$ and $e^{-\frac{k\pi}{n}i}$, and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & \mu_{k,n} \end{smallmatrix}\right)^n = 1$. Therefore

$$\begin{pmatrix} q_n(\mu_{k,n}) \\ q_{n+1}(\mu_{k,n}) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \mu_{k,n} \end{pmatrix}^n \begin{pmatrix} q_0(\mu_{k,n}) \\ q_1(\mu_{k,n}) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and $q_n(\mu_{k,n}) = 0$. Since the degree of q_n is $n-1$ these are all the roots of q_n. By induction one can see that $q_n(2) > q_{n-1}(2)$, and therefore $q_n(2) > 0 \; \forall n > 0$. Since $\mu_{1,n} < 2$ is the largest root of q_n, it follows that $q_n(\nu) > 0 \; \forall \nu > \mu_{1,n}$ and especially $q_n(\lambda) > 0$ for $l > n$. Since $\lambda = \mu_{1,l}$ we have $q_l(\lambda) = 0$ and $q_{l+1}(\lambda) = \lambda q_l(\lambda) - q_{l-1}(\lambda) < 0$. Therefore the claim about p holds.

The only remaining cases are the Coxeter diagrams of types H_3 and H_4. With $\lambda = 2\cos\frac{\pi}{5}$ and therefore $\lambda^2 = \lambda + 1$ one calculates the two translation quivers (see Table 3 on page 56) and compares them to the ones in Appendix C. $\square$

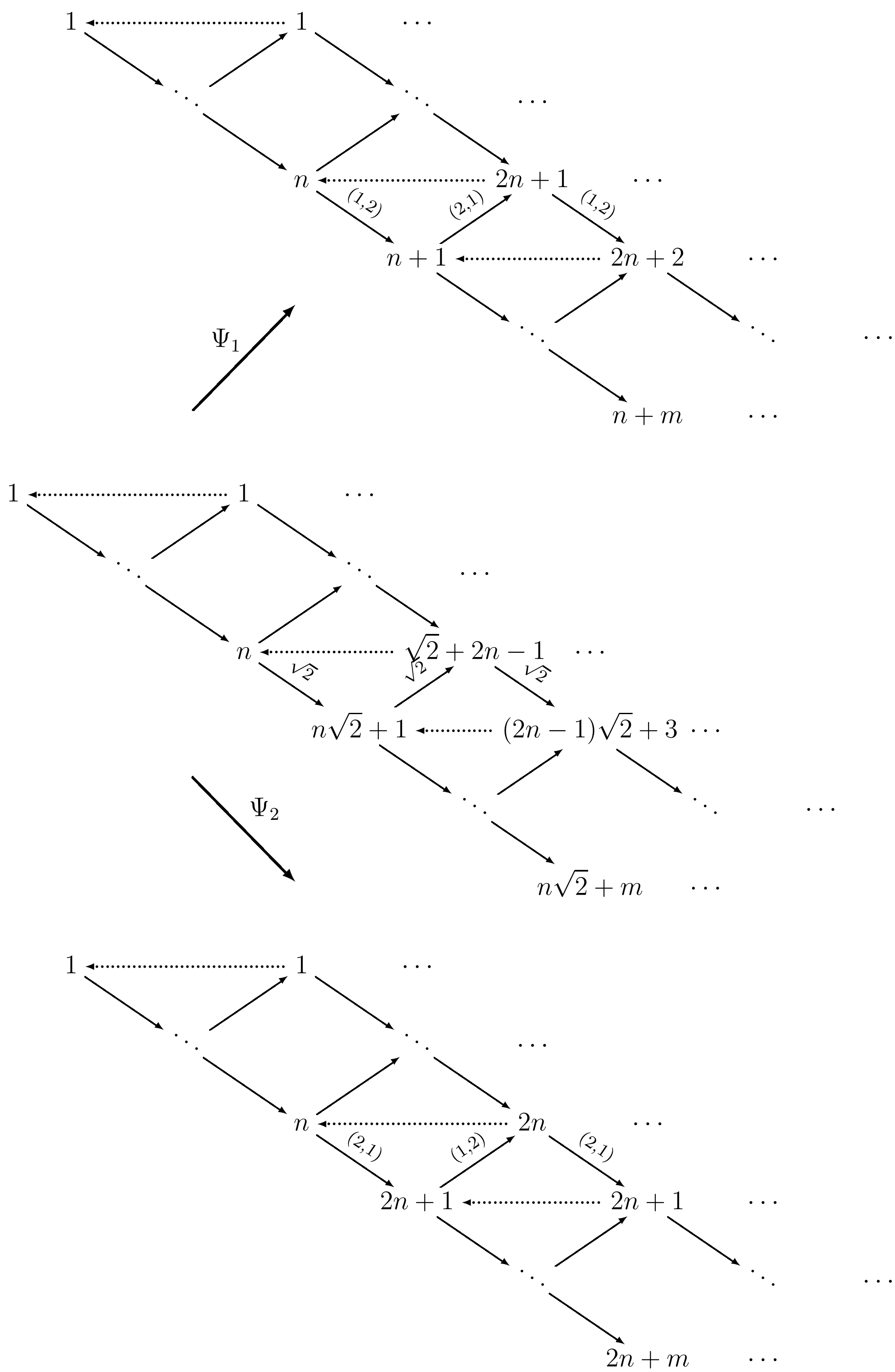


Table 2: Conversion of lengths in cases BC_n or F_4

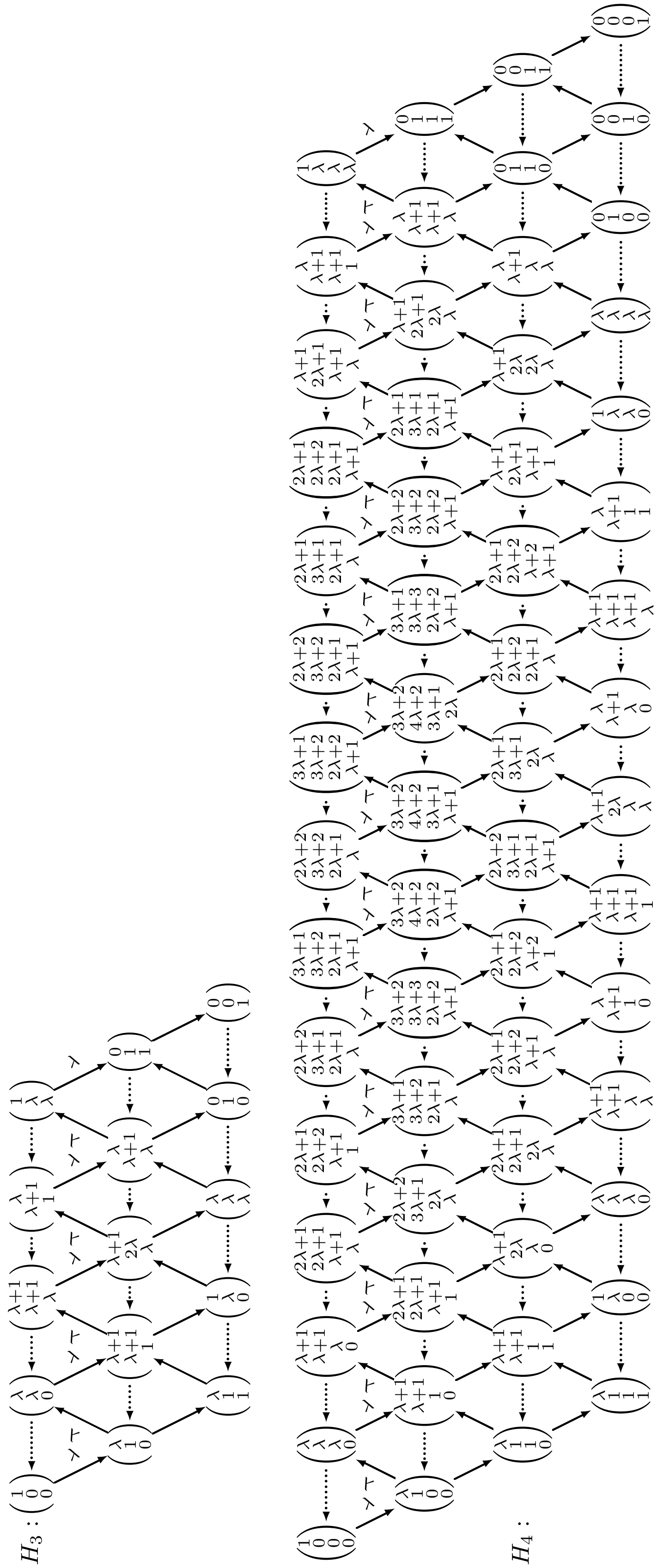


Table 3: Positive roots of Coxeter diagrams of type H_3 and H_4

9.11 Remark. The proof also shows the claim of Theorem 8.10 in the case that the species only contains dimension sequences $(1, 1, 1)$ and $(1, 2, 1, 2)$, which was originally only cited.

10 Coverings of strict τ-categories

In this section, we will introduce coverings of strict τ-categories the way Bongartz and Gabriel [3] did for what the call "Auslander categories". We will find out what coverings do on the module categories. This will help us show that coverings behave nicely on almost split sequences (10.3, 10.4 and 10.6).

10.1 Definition. Let $\mathcal{U}$ and $\mathcal{C}$ be Krull-Schmidt categories. For simplicity assume that $\mathcal{U}$ and $\mathcal{C}$ are skeletal, i.e. every isomorphism class contains exactly one object.

Following [3], we call an additive functor $\mathtt{F} : \mathcal{U} \longrightarrow \mathcal{C}$ a *covering* if it satisfies the following three conditions.

1. $\mathtt{F}$ is faithful
2. $\mathtt{F}(\mathrm{Ind}(\mathcal{U})) = \mathrm{Ind}(\mathcal{C})$
3. $\forall a, b \in \mathrm{Ind}(\mathcal{C})\ \forall u : \mathtt{F}(u) = b$

$$\mathcal{C}(a,b) = \coprod_{v\colon \mathtt{F}(v)=a} \mathtt{F}(\mathcal{U}(v,u)) \qquad \mathcal{C}(b,a) = \coprod_{v\colon \mathtt{F}(v)=a} \mathtt{F}(\mathcal{U}(u,v))$$

In this section $\mathtt{F}$ is always assumed to be a covering $\mathcal{U} \longrightarrow \mathcal{C}$. Obviously, it is sufficient to know $\mathtt{F}$ on $\mathrm{Ind}(\mathcal{U})$.

The following two functors between the module categories are induced by $\mathtt{F}$:

$$\begin{aligned} \mathtt{F}^\star &: \mathcal{C}\text{-Mod} \longrightarrow \mathcal{U}\text{-Mod} : M \longmapsto M \circ F \text{ and} \\ \mathtt{F}_\star &: \mathcal{U}\text{-Mod} \longrightarrow \mathcal{C}\text{-Mod} : M \longmapsto [c \longmapsto \coprod_{u\colon \mathtt{F}(u)=c} M(u)]. \end{aligned}$$

As a sequence in a module category is exact if it is exact in every component, the functors $\mathtt{F}^\star$ and $\mathtt{F}_\star$ are exact and reflect exactness.

Finitely generated (indecomposable) projective $\mathcal{U}$-modules are mapped to finitely generated (indecomposable) projective $\mathcal{C}$-modules by $\mathtt{F}_\star$:

$$\mathtt{F}_\star(P_u) = [c \longmapsto \coprod_{v\colon \mathtt{F}(v)=c} P_u^v] = P_{F(u)}$$

Hence $\mathtt{F}_\star$ also defines two restricted functors $\mathcal{U}\text{-Mod}_{\mathrm{fg}} \longrightarrow \mathcal{C}\text{-Mod}_{\mathrm{fg}}$ and $\mathcal{U}\text{-mod} \longrightarrow \mathcal{C}\text{-mod}$.

10.2 Lemma. *A morphism is in the radical of $\mathcal{U}$ if and only if its image is in the radical of $\mathcal{C}$.*

Proof. Without loss of generality we may assume the morphism in question to be $\alpha : u \longrightarrow v$ with u and v indecomposable.

If $\alpha \notin \mathcal{J}_{\mathcal{U}}$ then α is an isomorphism. Then so is $\mathtt{F}(\alpha)$ and therefore $\mathtt{F}(\alpha) \notin \mathcal{J}_{\mathcal{C}}$.

If $\mathtt{F}(\alpha) \notin \mathcal{J}_{\mathcal{C}}$, then $\mathtt{F}(\alpha)$ is invertible. Let $\beta \in \mathcal{C}$ be such that $\mathtt{F}(\alpha)\beta = 1$. By definition of a covering β can be written as

$$\beta = \sum_{\mathtt{F}(\tilde{u})=\mathtt{F}(u)} \mathtt{F}(\beta_{\tilde{u}}) \text{ with } \beta_{\tilde{u}} : v \longrightarrow \tilde{u}$$

Now $\sum_{\mathtt{F}(\tilde{u})=u} \mathtt{F}(\alpha\beta_{\tilde{u}}) = \mathtt{F}(1_u)$, and therefore, because the sum in the definition of coverings is direct, $\alpha\beta_u = 1_u$ and all other $\alpha\beta_{\tilde{u}}$ are zero. Similarly α has a left inverse. Therefore $\alpha \notin \mathcal{J}_{\mathcal{U}}$. $\square$

In particular, $\mathtt{F}_\star$ maps simple $\mathcal{U}$-modules S_u to the corresponding simple $\mathcal{C}$-modules $S_{\mathtt{F}(u)}$.

10.3 Theorem. *A sequence $u \xrightarrow{\nu} v \xrightarrow{\mu} w$ in $\mathcal{U}$ is a right almost split sequence if and only if its image $\mathtt{F}(u) \xrightarrow{\mathtt{F}(\nu)} \mathtt{F}(v) \xrightarrow{\mathtt{F}(\mu)} \mathtt{F}(w)$ is one.*

Proof. By Lemma 10.2, $\nu \in \mathcal{J}_{\mathcal{U}} \iff \mathtt{F}(\nu) \in \mathcal{J}_{\mathcal{C}}$ and $\mu \in \mathcal{J}_{\mathcal{U}} \iff \mathtt{F}(\mu) \in \mathcal{J}_{\mathcal{C}}$. Therefore assume that all these morphisms are in the radical. By Theorem 3.13, $u \xrightarrow{\nu} v \xrightarrow{\mu} w$ is a right almost split sequence if and only if $P_u \xrightarrow{P_\nu} P_v \xrightarrow{P_\mu} P_w \xrightarrow{\pi} S_w$ is exact. This is equivalent to the exactness of $\mathtt{F}_\star(P_u) \xrightarrow{\mathtt{F}_\star(P_\nu)} \mathtt{F}_\star(P_v) \xrightarrow{\mathtt{F}_\star(P_\mu)} \mathtt{F}_\star(P_w) \xrightarrow{\mathtt{F}_\star(\pi)} \mathtt{F}_\star(S_w)$. But, since $\mathtt{F}_\star(P_u) = P_{\mathtt{F}(u)}$ and $\mathtt{F}_\star(S_u) = P_{\mathtt{S}(u)}$, this is equivalent to $\mathtt{F}(u) \xrightarrow{\mathtt{F}(\nu)} \mathtt{F}(v) \xrightarrow{\mathtt{F}(\mu)} \mathtt{F}(w)$ being a right almost split sequence. $\square$

10.4 Corollary. *A sequence $u \xrightarrow{\nu} v \xrightarrow{\mu} w$ in $\mathcal{U}$ is an almost split sequence if and only if $\mathtt{F}(u) \xrightarrow{\mathtt{F}(\nu)} \mathtt{F}(v) \xrightarrow{\mathtt{F}(\mu)} \mathtt{F}(w)$ is one.*

10.5 Lemma. *If $\mathcal{C}$ is strongly noetherian, then so is $\mathcal{U}$.*

Proof. Assume $\mathcal{C}$ is strongly noetherian, and let $u \in \mathcal{O}\!\mathit{b}\,\mathcal{U}$ and $M \leq P_u$. Then we have a short exact sequence $M \xrightarrow{\iota} P_u \xrightarrow{\pi} C$ in $\mathcal{U}$-Mod. By exactness of $\mathtt{F}_\star$, $\mathtt{F}_\star(M) \xrightarrow{\mathtt{F}_\star(\iota)} P_{\mathtt{F}(u)} \xrightarrow{\mathtt{F}_\star(\pi)} \mathtt{F}_\star(C)$ is exact in $\mathcal{C}$-Mod, and, since $\mathcal{C}$ is strongly noetherian the sequence actually lies in $\mathcal{C}$-mod. Therefore there is an epimorphism $\gamma : P_x \longrightarrow \mathtt{F}_\star(M)$ for some $x \in \mathcal{O}\!\mathit{b}\,\mathcal{C}$. Now $\gamma\mathtt{F}_\star(\iota) \in \mathcal{C}(x, \mathtt{F}(u))$, and therefore $\gamma\mathtt{F}_\star(\iota) = \sum \mathtt{F}(\delta_{\tilde{x}})$, $\delta_{\tilde{x}} : \tilde{x} \longrightarrow u$. Since

$\sum \mathtt{F}_\star(\delta_{\tilde{x}}\pi) = \sum \mathtt{F}(\delta_{\tilde{x}})\mathtt{F}_\star(\pi) = 0$, all $\delta_{\tilde{x}}\pi$ must be zero. Therefore, every $\delta_{\tilde{x}}$ factors through ι, say $\delta_{\tilde{x}} = \gamma_{\tilde{x}}\iota$. Now clearly

$$\bigoplus_{\tilde{x}:\gamma_{\tilde{x}}\neq 0} P_{\tilde{x}} \xrightarrow{(\gamma_{\tilde{x}})} M,$$

and therefore M is finitely generated. With Lemma 3.6 it follows that $\mathcal{U}$ is strongly noetherian. □

10.6 Theorem. *Let $\mathcal{C}$ be strongly noetherian, $\tau c \rightarrowtail \vartheta c \longrightarrow c$ a right almost split sequence in $\mathcal{C}$ and $u \in \mathcal{U}$ with $\mathtt{F}(u) = c$. Then u has a right almost-split sequence $\tau u \rightarrowtail \vartheta u \longrightarrow u$ and*

$$\mathtt{F}(\tau u \rightarrowtail \vartheta u \longrightarrow u) \cong \tau c \rightarrowtail \vartheta c \longrightarrow c.$$

Proof. By assumption, $P_{\tau c} \rightarrowtail P_{\vartheta c} \longrightarrow P_c \twoheadrightarrow S_c$ is exact. Therefore, so is

$$\mathtt{F}^\star(P_{\tau c}) \rightarrowtail \mathtt{F}^\star(P_{\vartheta c}) \longrightarrow \mathtt{F}^\star(P_c) \twoheadrightarrow \mathtt{F}^\star(S_c).$$

Since S_u is a direct summand of $\mathtt{F}^\star(S_c) = \coprod_{\mathtt{F}(x)=c} S_x$, S_u has a minimal projective resolution of length two in $\mathcal{U}$-Mod, which is in $\mathcal{U}$-mod as $\mathcal{U}$ is strongly noetherian. Therefore u has a right almost split sequence.

$$\mathtt{F}(\tau u \rightarrowtail \vartheta u \longrightarrow u) \cong \tau c \rightarrowtail \vartheta c \longrightarrow c$$

follows from the uniqueness of right almost split sequences. □

10.7 Lemma. *Let $\mathcal{C}$ be strongly noetherian. If $\mathcal{C}$ has kernels then so does $\mathcal{U}$.*

Proof. Let $\alpha : u \longrightarrow v$ be a morphism in $\mathcal{U}$. By assumption, we can find κ such that $k \overset{\kappa}{\rightarrowtail} \mathtt{F}(u) \xrightarrow{\mathtt{F}(\alpha)} \mathtt{F}(v)$ is exact. Then $P_k \overset{P_\kappa}{\rightarrowtail} P_{\mathtt{F}(u)} \xrightarrow{P_{\mathtt{F}(\alpha)}} P_{\mathtt{F}(v)}$ is also exact. Applying $\mathtt{F}^\star$ we get the following commutative diagram with exact rows:

$$\begin{tikzcd}
\mathtt{F}^\star(P_k) \arrow[r, tail, "\mathtt{F}^\star(P_\kappa)"] \arrow[d, equal] & \mathtt{F}^\star(P_{\mathtt{F}(u)}) \arrow[r, "\mathtt{F}^\star(P_{\mathtt{F}(\alpha)})"] \arrow[d, equal] & \mathtt{F}^\star(P_{\mathtt{F}(v)}) \arrow[d, equal] \\
\coprod_{\mathtt{F}(\widetilde{k})=k} P_{\widetilde{k}} \arrow[r, tail] & \coprod_{\mathtt{F}(\widetilde{u})=\mathtt{F}(u)} P_{\widetilde{u}} \arrow[r] & \coprod_{\mathtt{F}(\widetilde{v})=\mathtt{F}(v)} P_{\widetilde{v}} \\
K \arrow[u] \arrow[r, tail] & P_u \arrow[u, "\text{split}"'] \arrow[r, "P_\alpha"] \arrow[ur, phantom, "Q"] & P_v \arrow[u, "\text{split}"']
\end{tikzcd}$$

Then the kernel morphism also splits and therefore K is projective. Also, K is finitely generated, since it is a submodule of P_u, so $K = P_l$ for some $l \in \mathcal{O}b\,\mathcal{U}$. Then l is the kernel of α. □

11 Construction of indecomposing coverings

In this section, we will restrict ourselves to indecomposing coverings, which are coverings that do not split arrows in Auslander-Reiten quivers (see Example 2 in Section 12 to understand this intuitive description). We will construct such coverings similar to constructing coverings of topological spaces. Especially we will find a universal covering.

Let $\mathcal{C}$ be a graded strict τ-category with a fixed finite grading. This means the $\mathcal{J}^{(i)} = \mathcal{J}^i/\mathcal{J}^{i+1}$ are subcategories of $\mathcal{C}$ in such a way that the composition satisfies $\mathcal{J}^{(i)}(a,b)\mathcal{J}^{(j)}(b,c) \subset \mathcal{J}^{(i+j)}(a,c)$, and $\mathcal{J}^n = 0$ for some n. We will further assume that $\mathcal{C}$ is connected (see Definition 11.2 below).

There are always coverings identifying two or more exact copies of $\mathcal{C}$ with another. To exclude that (uninteresting) case, we will, from now on, assume $\mathcal{U}$ to be connected.

11.1 Definition. We call a covering *indecomposing*, if, for any $u, v \in \mathrm{Ind}(\mathcal{U})$ the following implication holds.

$$\mathcal{J}^{(1)}(u,v) \neq 0 \Rightarrow \mathcal{J}^{(1)}(\mathtt{F}(u), \mathtt{F}(v)) = \mathtt{F}(\mathcal{J}^{(1)}(u,v))$$

See Section 12, Example 2 for a covering, which is not indecomposing. However, if $\mathcal{C} = A$-mod for some representation-finite artinian ring A, it will turn out that all covers are indecomposing (Corollary 14.4).

Let $\widetilde{\mathcal{W}}$ be the free category ($\widetilde{\mathcal{W}}$ is not preadditive) with $\mathcal{O}\!\mathit{b}\ \mathcal{W} = \mathrm{Ind}(\mathcal{C})$ generated by the morphisms $a \xrightarrow{(a,b)} b$ and $b \xrightarrow{\widehat{(a,b)}} a$ whenever $\mathcal{J}^{(1)}(a,b) \neq 0$. This means that $\widetilde{\mathcal{W}}$ consists of all paths in the Auslander-Reiten quiver of $\mathcal{C}$. Let

$$\mathcal{W} = \widetilde{\mathcal{W}}/\big\langle (a,b)\widehat{(a,b)} \sim 1_a,\ \widehat{(a,b)}(a,b) \sim 1_b,\ (\tau a, b_1)(b_1, a) \sim (\tau a, b_2)(b_2, a)\big\rangle$$

We call the elements of $\mathcal{W}$ *walks* in $\mathcal{C}$. The relations factored out mean, that walking "there and back again" (by the same path) is as good as not walking at all, and that it does not affect the walk which way one takes around an almost split sequence in the Auslander-Reiten quiver.

Fix an $x \in \mathrm{Ind}(\mathcal{C})$.

11.2 Definition. The strict τ-category $\mathcal{C}$ is called connected, if $\mathcal{W}(a,b) \neq \emptyset$ for all a, $b \in \mathrm{Ind}(\mathcal{C})$.

The *fundamental group* of $\mathcal{C}$ (with respect to x) is $\mathcal{W}(x,x)$ and will be called G. If the fundamental group is trivial, $\mathcal{C}$ is called *simply connected*.

Since $\mathcal{C}$ is assumed to be connected, G is independent of the choice of x.

Let S be a subgroup of G. We will now construct an indecomposing covering $\mathcal{U}$ of $\mathcal{C}$ with fundamental group S. Set

$$\begin{aligned} \operatorname{Ind}(\mathcal{U}) &= \{Sw \mid w \in \mathcal{W}(x,y) \text{ for some } y \in \operatorname{Ind}(\mathcal{C})\} \\ \mathcal{U}(vS, wS) &= \big\langle \varphi_1 \cdots \varphi_n \mid \varphi_i \in \mathcal{J}^{(1)}(a_{i-1}, a_i) \text{ and} \\ &\qquad (a_0, a_1) \cdots (a_{n-1}, a_n) \in v^{-1}Sw \text{ in } \mathcal{W}\big\rangle \end{aligned}$$

and define $\mathtt{F} : \operatorname{Ind}(\mathcal{U}) \longrightarrow \operatorname{Ind}(\mathcal{C})$ by Sw is mapped to the final object of the walk w, and $\mathtt{F}$ is the identity on morphisms.

11.3 Theorem. *For any subgroup S of the fundamental group of $\mathcal{C}$, the covering $\mathtt{F} : \mathcal{U} \longrightarrow \mathcal{C}$ constructed above is a covering, such that the fundamental group of $\mathcal{U}$ is S.*

Proof. It is clear that $\mathtt{F}$ is faithful and that $\mathtt{F}$ is surjective on objects. We have to check, that $\mathcal{C}(a, \mathtt{F}(u)) = \coprod_{v\colon \mathtt{F}(v)=a} \mathtt{F}(\mathcal{U}(v,u))$. Then the other equation is dual, and therefore it will follow that $\mathtt{F}$ is a covering.

It is clear that the $\mathtt{F}(\mathcal{U}(v,u))$ with $\mathtt{F}(v) = a$ generate $\mathcal{C}(a, \mathtt{F}(u))$. Assume $\sum \varphi_v = 0$ with $\varphi_v \in \mathcal{U}(v,u)$ not all zero. Since $\mathcal{C}$ is graded we may assume that all the φ_v are in the same power of the radical, say $\varphi_v \in \mathcal{J}^{(n)}\ \forall v$ and we may assume this power n to be minimal. We may assume $n > 0$, since otherwise all φ_v with $v \neq u$ would have to be zero, leaving just maximally one summand, which then would have to be zero by the equation. Hence the φ_v factor through the components of $\vartheta\mathtt{F}(u)$. We will denote the indecomposable direct summands of $\vartheta\mathtt{F}(u)$ by x_i, and the components of ν and μ will be called ν_i and μ_i respectively. They can be assumed to be chosen in such a way, that ν_i and μ_i are in $\mathcal{J}^{(1)}$. If $\varphi_v = \sum_i \widetilde{\varphi}_{v,i}\, \mu_i$ then

$$\Big(\sum_v \widetilde{\varphi}_{v,i}\Big)_i : a \longrightarrow \bigoplus_i x_i = \vartheta\mathtt{F}(u)$$

factors through $\nu = \ker \mu$. Let α be such that $\sum_v \widetilde{\varphi}_{v,i} = \alpha\nu_i$. We can decompose $\alpha = \sum_v \widetilde{\alpha}_v$ with $\widetilde{\alpha}_v \in \mathcal{U}(v, \tau u)$, where $\tau u = u(\widehat{x_i, \mathtt{F}(u)})(\widehat{\tau\mathtt{F}(u), x_i})$ is well defined by the relations in $\mathcal{W}$. By minimality of n, we know that $\widetilde{\varphi}_{v,i} = \nu_i\alpha_v$. Hence $\varphi_v = \sum_i \widetilde{\varphi}_{v,i}\mu_i = \sum_i \alpha_v\nu_i\mu_i = 0\ \forall v$.

Finally, by construction of $\mathcal{U}$, we have $G_{\mathcal{U}} = \{w \in G | Sw = S\} = S$. $\square$

11.4 Theorem. *Every indecomposing covering is obtained in the above construction.*

Proof. Let $\mathtt{F} : \mathcal{U} \longrightarrow \mathcal{C}$ be an indecomposing covering. Then $\mathtt{F}$ induces a bijection

$$\bigcup_{v \in \mathrm{Ind}(\mathcal{U})} \mathcal{W}_{\mathcal{U}}(u, v) \longrightarrow \bigcup_{a \in \mathrm{Ind}(\mathcal{C})} \mathcal{W}_{\mathcal{C}}(\mathtt{F}(u), a)$$

and hence an injection $G_{\mathcal{U}} \rightarrowtail G_{\mathcal{C}}$. Let $\widetilde{\mathcal{U}}$ be the covering obtained for $S = G_{\mathcal{U}}$ in the above construction. Define

$$\tilde{\mathtt{F}} : \widetilde{\mathcal{U}} \longrightarrow \mathcal{U} : Sw \longmapsto \text{ final object of } w \text{ interpreted as walk in } \mathrm{Ind}(\mathcal{U})$$

by $\tilde{\mathtt{F}}(\alpha) = \alpha$ for $\alpha \in \mathcal{J}^{(i)}, i \in \{0, 1\}$.

This is well defined and injective as the functor induces a bijection on the fundamental groups. Therefore $\tilde{\mathtt{F}}$ is full, faithful and dense, hence the two coverings are isomorphic. □

11.5 Corollary. *The indecomposing coverings of $\mathcal{C}$ form a lattice which is anti-isomorphic to the lattice of subgroups of the fundamental group.*

Proof. Obviously one covering constructed as above factors through another if and only if its fundamental group, regarded as subgroup of the fundamental group of $\mathcal{C}$, is included in the one of the other covering. □

11.6 Definition. The largest indecomposing covering (i.e. the one corresponding to the trivial subgroup and therefore factoring through every other indecomposing covering) will be called the *universal (indecomposing) covering.*

The universal indecomposing covering is the only indecomposing covering with $\mathcal{U}$ simply connected.

12 Examples of coverings

Let us now pause for a moment and look at a few examples. We want to get an intuitive idea of what coverings and the universal covering are in the pictures of Auslander-Reiten quivers.

1. Let $\mathcal{C} = F[x]/(x^2)$-mod $= FQ/(\alpha^2)$-mod with $Q = \begin{tikzcd}\circ \arrow[loop right, "\alpha"]\end{tikzcd}$. Then its Auslander-Reiten quiver $\Gamma_{\mathcal{C}}$ is

$$\begin{tikzcd}
 & {[P]} \arrow[dr] & \\
S \arrow[ur] & & S \arrow[ll, dotted]
\end{tikzcd}$$

Here P is the indecomposable projective injective module and S is the simple one.

(a) Let $\mathcal{U} = F\widetilde{Q}/(\gamma\beta, \beta\gamma)$-mod with $\widetilde{Q} = \begin{tikzcd}1 \arrow[r, bend right, "\beta"'] & 2 \arrow[l, bend right, "\gamma"']\end{tikzcd}$.

Then the corresponding Auslander-Reiten quiver is (the P_i are the projective modules, the S_i the corresponding simple ones)

$$\Gamma_{\mathcal{U}}: \begin{tikzcd}
 & {[P_2]} \arrow[dr] & & {[P_1]} \arrow[dr] & \\
S_1 \arrow[ur] & & S_2 \arrow[ur] \arrow[ll, dotted] & & S_1 \arrow[ll, dotted]
\end{tikzcd}$$

Then

$$\mathtt{F} : \mathcal{U} \longrightarrow \mathcal{C} : \begin{tikzcd}V_1 \arrow[r, bend right, "V_\beta"'] & V_2 \arrow[l, bend right, "V_\gamma"']\end{tikzcd} \longmapsto \begin{tikzcd}V \arrow[loop above, "V_\alpha"]\end{tikzcd}, V = V_1 \bigoplus V_2, V_\alpha = \begin{pmatrix} 0 & V_\gamma \\ V_\beta & 0 \end{pmatrix}$$

is an indecomposing covering of $\mathcal{C}$.

(b) Let $A_\infty^\infty = \cdots \longrightarrow i-1 \xrightarrow{\alpha_{i-1}} i \xrightarrow{\alpha_i} i+1 \longrightarrow \cdots$ and let $\mathcal{U} \subset FA_\infty^\infty/(\alpha_{i-1}\alpha_i \mid i \in \mathbb{Z})$-mod be the full subcategory defined by $\mathcal{O}b\,\mathcal{U} = \{V \mid \sum \dim_F V_i < \infty\}$. Then the Auslander-Reiten quiver is

$$\Gamma_{\mathcal{U}}: \cdots \begin{tikzcd}
 & {[P_{i-1}]} \arrow[dr] & & {[P_i]} \arrow[dr] & & {[P_{i+1}]} & \cdots \\
S_{i-2} \arrow[ur] & & S_{i-1} \arrow[ur] \arrow[ll, dotted] & & S_i \arrow[ur] \arrow[ll, dotted] & &
\end{tikzcd}$$

Define $\mathtt{F} : \mathcal{U} \longrightarrow \mathcal{C}$ similarly to the previous example. Then $\mathtt{F}$ is an indecomposing covering und $\mathcal{U}$ is simply connected, so $\mathtt{F}$ is the universal indecomposing covering.

2. Let $\mathcal{C}$ be the preprojective component of FQ, where Q is the Kronecker-Quiver $1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$. Then the Auslander-Reiten quiver of $\mathcal{C}$ is

$$\Gamma_{\mathcal{C}}: \quad \begin{tikzcd}
M_1 \arrow[dr, "{(2,2)}"] & & M_3 \arrow[ll, dotted] \arrow[dr, "{(2,2)}"] & & M_5 \arrow[ll, dotted] \arrow[dr, "{(2,2)}"] & & \\
& M_2 \arrow[ur, "{(2,2)}"] & & M_4 \arrow[ll, dotted] \arrow[ur, "{(2,2)}"] & & M_6 \arrow[ll, dotted] & \cdots
\end{tikzcd}$$

Let $\mathcal{U}$ the preprojective component $F\widetilde{A_3}$ with $\widetilde{A_3} = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & 2 \\ 3 \arrow[r] \arrow[ur] & 5 \end{tikzcd}$. Then its Auslander Reiten quiver is

$$\Gamma_{\mathcal{U}}: \quad \begin{tikzcd}
N_1^1 \arrow[dr] \arrow[dddr] & & N_3^1 \arrow[ll, dotted] \arrow[dr] \arrow[dddr] & & N_5^1 \arrow[ll, dotted] \arrow[dr] \arrow[dddr] & & \\
& N_2^1 \arrow[ur] \arrow[dr] & & N_4^1 \arrow[ll, dotted] \arrow[ur] \arrow[dr] & & N_6^1 \arrow[ll, dotted] & \cdots \\
N_1^2 \arrow[ur] \arrow[dr] & & N_3^2 \arrow[ll, dotted] \arrow[ur] \arrow[dr] & & N_5^2 \arrow[ll, dotted] \arrow[ur] \arrow[dr] & & \\
& N_2^2 \arrow[uuur] \arrow[ur] & & N_4^2 \arrow[ll, dotted] \arrow[uuur] \arrow[ur] & & N_6^2 \arrow[ll, dotted] &
\end{tikzcd}$$

There is a covering $\mathtt{F} : \mathcal{U} \longrightarrow \mathcal{C}$ with $\mathtt{F}(N_i^j) = M_i$.

Since this covering splits the $\mathcal{J}^{(1)}(M_i, M_{i+1})$ into the two vector spaces $\mathcal{J}^{(1)}(N_i^j, N_{i+1}^j) \bigoplus \mathcal{J}^{(1)}(N_i^j, N_{i+1}^{2-j})$, this is an example of a covering that is not indecomposing. In fact, $\mathcal{C}$ is already simply connected, and therefore does not admit any nontrivial indecomposing coverings.

13 Coverings of the module categories of representation-finite artinian rings

In this section, we will apply coverings, and especially universal coverings, to the module categories of representation-finite artinian rings. We will see that is possible to find, in the possibly (and probably) infinite universal covering, subcategories, which are again the module categories of representation-finite artinian rings.

Throughout this section we assume A to be a representation-finite artinian ring. By 3.20, this implies that A-mod is a strict τ-category.

13.1 Definition. Let $\mathcal{C}$ be a Krull-Schmidt category. The *associated graded category* $\mathtt{Gr}(\mathcal{C})$ is defined by $\mathcal{O}b\ \mathtt{Gr}(\mathcal{C}) = \mathcal{O}b\,\mathcal{C}$ and $\mathtt{Gr}(\mathcal{C})(a,b) = \coprod \mathcal{J}^{(n)}(a,b)$ with the obvious composition.

For the remainder of this section, let $\mathcal{C}$ be $\mathtt{Gr}(A\text{-mod})$ and $\mathtt{F} : \mathcal{U} \longrightarrow \mathcal{C}$ a covering.

13.2 Lemma. *A $\mathcal{C}$-module is finitely generated if and only if it has finite length.*

Proof. It is obviously sufficient to show that P_m has finite length for any indecompsable A-module m. Let n be another indecomposable A-module. The number of composition factors of P_m isomorphic to the simple module S_n is $\dim_{\mathrm{End}(P_n)} \mathrm{Hom}(P_n, P_m) = \dim_{\mathrm{End}(n)} \mathrm{Hom}(n, m)$ which is finite, since A-mod is a strict τ-category. Adding up these finitely many finite numbers we find that the length of P_m is finite. $\square$

13.3 Corollary. *The category $\mathcal{C}$ is strongly noetherian.*

Let R be the ring of matrices $(\mathcal{U}(p,q))_{p,q}$, where p and q run through the indecomposable projective objects in $\mathcal{U}$. This is a ring since, for any p, the sets $\{q \mid \mathcal{U}(p,q) \neq 0\}$ and $\{q \mid \mathcal{U}(q,p) \neq 0\}$ are finite and hence the matrices in R are row- and column-finite.

13.4 Theorem. *Let R-mod$_{\mathrm{fl}}$ be the category of R-modules of finite length. Then $\mathcal{U} \approx R\text{-mod}_{\mathrm{fl}}$.*

Proof. By Lemma 10.7 and its dual $\mathcal{U}$ has kernels and cokernels. Since $\mathcal{C}$ has sufficiently many projectives so does $\mathcal{U}$. By construction of R the full subcategories of projective objects in U and in R-mod$_{\mathrm{fl}}$ are equivalent. Therefore the claim of the theorem holds. $\square$

The ring R is locally representation-finite in the following sense: for any indecomposable projective R-module p there are only finitely many nonisomorphic indecomposable modules m with $\operatorname{Hom}(p, m) \neq 0$. The following construction works for an arbitrary locally representation-finite R with $R\text{-mod}_{\mathrm{fl}}$ Krull-Schmidt:

13.5 Construction. Let R be locally representation-finite and $m \in R\text{-mod}_{\mathrm{fl}}$. Let $P = \{p \in R\text{-proj} \mid \operatorname{Hom}_R(p, m) = 0\}$ be the set of all projective modules without homomorphisms to m. Then P contains almost all indecomposable projective modules.

Clearly $I = \operatorname{Tr} P$ is an ideal of R, and R/I is representation-finite because

$$R/I\text{-mod} = \{m \in R\text{-mod} \mid \text{ no summand of the projective cover of } m \text{ is in } P\}.$$

For a set of modules M, $\mathcal{G}en\, M$ denotes the full subcategory of the module category, whose objects are the epimorphic images of direct sums of elements of M.

13.6 Lemma. *In $R\text{-mod}_{\mathrm{fl}}$, $(R/I\text{-mod}, \mathcal{G}en\, P)$ defines a torsion pair (see [11] for the definition).*

Proof. For any $m \in R\text{-mod}_{\mathrm{fl}}$, we have $\operatorname{Tr}_P(m) \rightarrowtail m \twoheadrightarrow m/Im$. Further, since the elements of P are projective, $\operatorname{Hom}_R(\mathcal{G}en\ P,\ R/I\text{-mod}) = 0$. $\square$

13.7 Corollary. *If $R\text{-mod}_{\mathrm{fl}}$ is simply connected, then so is $R/I\text{-mod}$ (if it is connected at all).*

For a finite set of indecomposable R-modules $\{m_i\}$, choose m as the direct sum of all indecomposable modules n, such that there are nonzero morphisms $m_i \longrightarrow n \longrightarrow m_j$. Then the m_i are obviously R/I-modules and $\operatorname{Hom}_R(m_i, m_j) = \operatorname{Hom}_{R/I}(m_i, m_j)$. Therefore the Auslander-Reiten-Quiver of $R\text{-mod}_{\mathrm{fl}}$ is "locally isomorphic" to the Auslander-Reiten-Quiver of a representation-finite ring. Moreover, if R is constructed as above from the universal covering of a representation-finite artinian ring, then the module category of the latter ring is simply connected.

13.8 Example. Let $\mathcal{U}$ be as in Section 12, Example 1b.

1. Let $P = \{P_i \mid i \in \mathbb{Z} \setminus \{0, 1\}\}$. Then $R/I \cong FA_2$ ($A_2 = \circ \longrightarrow \circ$) ($\cong \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$), and the Auslander-Reiten quiver of R/I is the following:

$$\Gamma_{R/I\text{-mod}} : \begin{tikzcd} & {[P_1]} \arrow[dr] & \\ {[S_0} \arrow[ur] & & {S_1]} \arrow[ll, dotted] \end{tikzcd}$$

2. Let $P = \{P_i \mid i \in \mathbb{Z} \setminus \{0,1,2\}\}$. Then $R/I \cong \left(\begin{smallmatrix} F & F & 0 \\ 0 & F & F \\ 0 & 0 & F \end{smallmatrix}\right)$, and the Auslander-Reiten quiver of R/I is the following:

$$\Gamma_{R/I\text{-mod}}: \begin{tikzcd} & {[P_1]} \arrow[dr] & & {[P_2]} \arrow[dr] & \\ {[S_0} \arrow[ur] & & S_1 \arrow[ll, dotted] \arrow[ur] & & S_2] \arrow[ll, dotted] \end{tikzcd}$$

14 Representation-finite artinian rings

In this very brief section we will see that two important properties of simply connected artinian rings can, by the use of coverings, be generalized to arbitrary representation-finite artinian rings resp. representation finite artinian rings with a graded module category.

14.1 Theorem. *Let A be a representation-finite artinian ring, $m, n \in A$-ind. Then $\mathcal{J}^{(1)}(m,n)$ is a dimension sequence bimodule.*

14.2 Theorem (Auslander-Reiten formula). *Let A be a representation-finite artinian ring such that A-mod is graded, $m, n \in A$-ind. Then*

$$\overline{\mathrm{Hom}}(n, \tau m)^{\mathrm{R}} \cong \mathrm{Ext}^1(m,n) \cong \underline{\mathrm{Hom}}(\tau^- n, m)^{\mathrm{L}}$$

14.3 Remark. Note that dualizing $\mathrm{Hom}(n, \tau m)$ makes sense, since it is a bimodule over the heads of the endomorphism rings because the module category is graded. This also shows that the formula, in this form, cannot be extended to rings with a module category which cannot be graded. However, the Auslander-Reiten quiver does not change when a category is replaced by the corresponding graded category.

Proof of both theorems. Both theorems are actually corollaries of the same theorems formulated for simply connected representation-finite artinian rings. Take the universal covering of the module category modulo the trace of all projective modules that are far enough away to have no homomorphisms to any of the objects in question (for the Auslander-Reiten formula these are an arbitrary preimage of one object, say m, the corresponding preimage of τm and all preimages of n which have homomorphisms or extensions with these). That is the module category of a representation-directed artinian ring, so the claims are already known in that category. □

14.4 Corollary (of 14.1). *Let $\mathcal{C} = A$-mod for a representation-finite artinian ring A. Then all coverings of $\mathcal{C}$ are indecomposing.*

Proof. By 14.1, every $\mathcal{J}^{(1)}(m,n)$ is a dimension sequence bimodule. If there was a decomposition $\mathcal{J}^{(1)}(m,n) = J_1 \oplus J_2$, then also $\mathcal{J}^{(1)}(m,n)^{i\mathrm{R}} = J_1^{i\mathrm{R}} \oplus J_2^{i\mathrm{R}}$. But a nonzero dimension sequence contains at least one "1", contradicting this decomposition. □

15 Gluing and ungluing of simple modules

The study of gluing and ungluing in this section serves two purposes: Firstly, it is an alternative way to simplify Auslander-Reiten quivers and possibly make them simply connected. Secondly, it puts a limit to the hope of finding some simple classification of representation-finite simply-connected artinian rings, as it provides a method of building arbitrarily large such rings.

15.1 Definition. A simple module which is neither injective nor projective nor a direct summand of the middle term of any almost split sequence will be called a *gluing point*.

15.2 Theorem (Gluing). *Let A be an artinian ring with a simple projective module S_P and a simple injective module S_I such that* $\mathrm{End}(S_P) \cong \mathrm{End}(S_I)$*. Then there is an artinian ring B with a gluing point S and a functor* $\mathtt{F} : A\text{-ind} \longrightarrow B\text{-ind}$ *such that the following two points hold:*

1. *The functor $\mathtt{F}$ is bijective on objects, except $\mathtt{F}(S_P) = \mathtt{F}(S_I) = S$.*
2. *The functor $\mathtt{F}$ is bijective on $\mathcal{J}^{(1)}$ morphisms, except on $\mathcal{J}^{(1)}(M, S_P)$ and on $\mathcal{J}^{(1)}(S_I, M)$.*

Proof. Write A as a matrix ring $A = (\mathrm{Hom}(P,Q))_{P,Q \text{ projective}}$. Let P_I be the projective cover of S_I. Let

$$B = (\mathrm{Hom}(P,Q) \bigoplus \mathrm{Hom}(P, S_I) \bigotimes_{\mathrm{End}(S_I)} \mathrm{Hom}(S_P, Q))_{P,Q \not\cong S_P \text{ projective}}$$
$$= (\mathrm{Hom}(P,Q) \bigoplus \delta_{P_I P} \mathrm{Hom}(S_P, Q))_{P,Q \not\cong S_P \text{ projective}}$$

and define

$$\mathtt{F} : \ ((M_P)_{P \text{ projective}} \longmapsto (M_P \bigoplus \delta_{P_I P} M_{S_P})_{P \not\cong S_P \text{ projective}}.$$

Then the claim of the theorem clearly holds. □

15.3 Corollary. *For two simply connected representation-finite artinian rings A_1 and A_2 and simple modules $S_I \in A_1$-inj and $S_P \in A_2$-proj such that* $\mathrm{End}(S_P) \cong \mathrm{End}(S_I)$ *there is a simply connected representation-finite artinian ring B with $\Gamma_B = \Gamma_{A_1} \biguplus \Gamma_{A_2} / (S_I = S_P)$.*

This tells us that the construction in 7.1 yields finite translation quivers, for quivers Q of arbitrary complexity, is the starting function s is chosen adequately. See Table 4 on page 71 for an example of this effect.

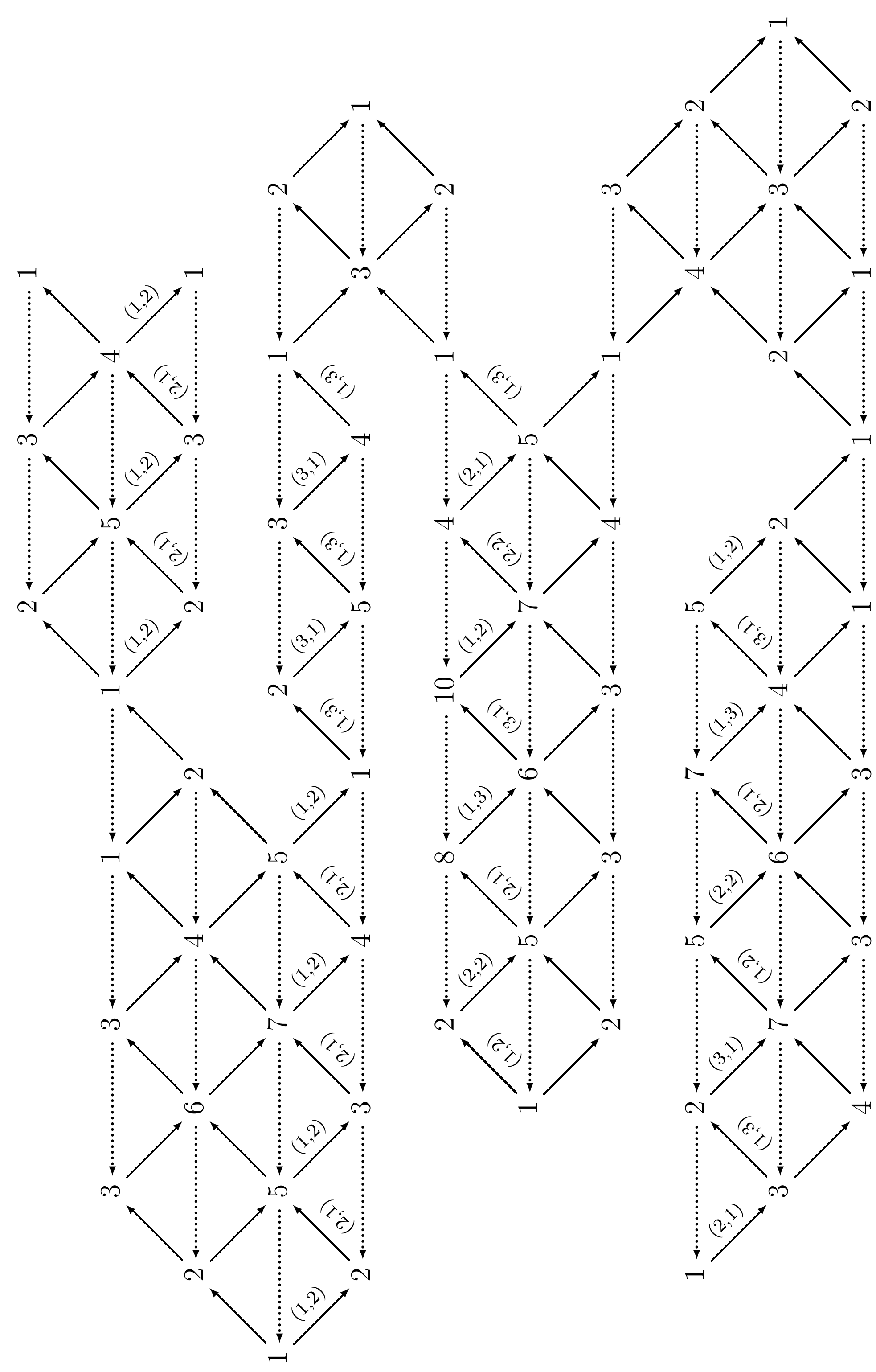


Table 4: Example of a large simply connected representation finite artinian ring obtained by gluing

15.4 Theorem (Ungluing). *Let A be an artinian ring with a gluing point S. Then there is an artinian ring B with a simple projective module S_P and a simple injective module S_I and a functor* $\mathtt{F}: B\text{-ind} \longrightarrow A\text{-ind}$ *such that the following two points hold:*

1. *The functor* $\mathtt{F}$ *is bijective on objects, except* $\mathtt{F}(S_P) = \mathtt{F}(S_I) = S$.
2. *The functor* $\mathtt{F}$ *is bijective on* $\mathcal{J}^{(1)}$ *morphisms except on* $\mathcal{J}^{(1)}(M, S_P)$ *and on* $\mathcal{J}^{(1)}(S_I, M)$.

Proof. Write A as a matrix ring $A = (\mathrm{Hom}(P,Q))_{P,Q \text{ projective}}$. Let P_I be the projective cover of S. Let

$$B = \begin{pmatrix} \{\varphi \in \mathrm{Hom}(P,Q) \mid \varphi \text{ doesn't factor through } S\}_{P,Q} & 0 \\ \mathrm{Hom}(S,Q)_Q & \mathrm{End}(S) \end{pmatrix}$$

Note, that $\{\varphi \in \mathrm{Hom}(P,Q) \mid \varphi \text{ doesn't factor through } S\} = \mathrm{Hom}(P,Q)$, if $P \ncong P_I$. Set

$$\mathtt{F}: \ M = \begin{pmatrix} (M_P)_{P \in A\text{-proj}} \\ M_S \end{pmatrix} \longmapsto (M_P \bigoplus \delta_{P_I P} M_S)_{P \text{ projective}}$$

Then the claim of the theorem clearly holds. □

15.5 Remark. Gluing and ungluing are trivial corollaries of Theorem 7.4, if its conditions are satisfied.

15.6 Remark. Let A be an artinian ring, which has a gluing point. Assume that the ring coming form A by ungluing this point is still connected. Then ungluing can be an alternative to covering in orderer to get a new ring with a similar representation theory to the one of A.

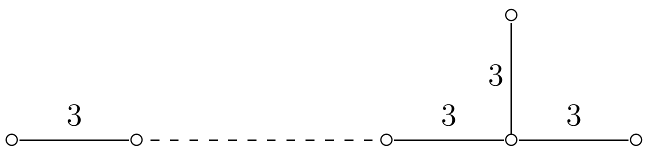
15.7 Example. Let $A = F[x]/(x^2)$ (see Section 12, Example 1). By ungluing one gets $\tilde{A} = FQ$ with $Q : \circ \longrightarrow \circ$.

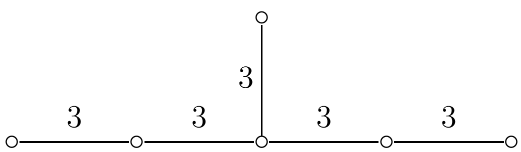
In 13.8 it was shown that one gets the same result by considering the universal cover of the module category and then going to the quotient module the trace of a set of projectives.

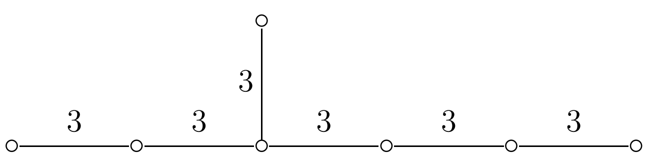
A The Coxeter diagrams

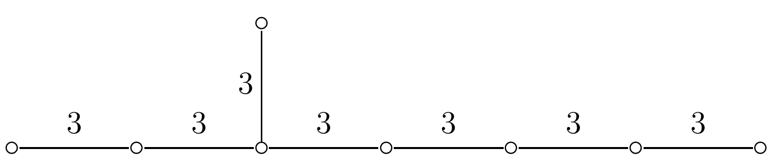
A_n : 

BC_n : 

D_n : 

E_6 : 

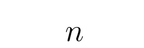
E_7 : 

E_8 : 

F_4 : 

H_3 : 

H_4 : 

I_n : 

B Computer program for combinatorial experiments

The following C++ program was written to produce, according to the description in 7.1, output like the Auslander-Reiten quivers given in Appendix C.

```
#include <stdio.h>
#include <string.h>
#include <iostream>

const int MAXPROJ = 16;
 //maximal number of projective modules
const int MAXWIDTH = 32;

const char BEGINDOC[] = "\\documentclass[a4paper,10pt]{article}\n\n\\
    usepackage{amsmath,amssymb,amsthm,diagrams}\n\n\\begin{document}\n$$\\
    begin{diagram}[w=1em,h=2em,tight,landscape]\n";
const char ENDDOC[] = "\\end{diagram}$$\n\\end{document}";

const char TEXCOMMAND[] = "latex quiver";
const char PSCOMMAND[] = "dvips quiver";
const char DVICOMMAND[] = "gv quiver.ps -seascape &";

int numproj = 1;
int s[MAXPROJ];
int d[MAXPROJ][MAXPROJ][MAXWIDTH];

class vector
// makes vectors behave like one would expect
{
private:
  int e[MAXPROJ];
public:
  friend vector operator +(vector v1, vector v2);
  friend vector operator *(int x, vector v);
  friend vector uvector(int i);
  friend int operator ==(vector v1, vector v2);
  friend void print(vector v, FILE* f);
  friend int positive(vector v);
};

int MenuChoice();
void AddProj();
void Reverse();
void Delay();
void ChDimSeq();
void DelProj();
void Calculate();

int main()
{
  s[0] = 0;
  Calculate();
  system(DVICOMMAND);
  while (1)
    {
      switch (MenuChoice())
        {
        case 0: return 1;
        case 1: AddProj(); break;
```

```
        case 2: Reverse(); break;
        case 3: Delay(); break;
        case 4: ChDimSeq(); break;
        case 5: DelProj(); break;
        }
    }
}

int MenuChoice()
{
  int n = -1;
  printf("\n0␣-␣Quit\n1␣-␣Add␣tau-orbit\n2␣-␣Reverse␣arrow\n3␣-␣Start␣tau-
      orbit␣later\n4␣-␣Change␣dimension␣sequence\n5␣-␣Delete␣tau-orbit\n");
  while ((n < 0) || (n > 5))
    {
      scanf("%d", &n);
    }
  return n;
}

void AddProj()
{
  if (numproj == MAXPROJ)
    {
      printf("Maximal␣number␣is␣allready␣reached.\n");
      return;
    }
  int n = -1;
  printf("Link␣new␣tau-orbit␣to␣");
  while ((n <= 0) || (n > numproj))
    {
      scanf("%d", &n);
    }
  n--;
  for (int i = 0; i <= numproj; i++)
    d[i][numproj][0] = d[numproj][i][0] = d[i][numproj][1] = d[numproj][i
        ][1] = 0;
  printf("Dimension␣sequence␣(0␣to␣finish)\n");
  for (int i = 0; i < MAXWIDTH; i++)
    {
      scanf("%d", &(d[n][numproj][i]));
      if (!d[n][numproj][i])
        {
          if (!i)
            {
              d[n][numproj][0] = 1;
              d[n][numproj][1] = 0;
            }
          i = MAXWIDTH;
        }
    }
  s[numproj] = s[n] + 1;
  numproj++;
  Calculate();
}

void Reverse()
{
  int n = -1, m = -1;
  printf("Reverse␣arrow␣from␣");
  while ((n <= 0) || (n > numproj))
    {
```

```
      scanf("%d", &n);
    }
  printf("                to ");
  while ((m <= 0) || (m > numproj))
    {
      scanf("%d", &m);
    }
  n--; m--;
  if (!(d[n][m][0]))
    printf("There is no arrow from %d to %d.", n+1, m+1);
  else
    {
      for (int k = 0; k < MAXWIDTH; k++)
        d[m][n][k] = d[n][m][k];
      d[n][m][0] = d[n][m][1] = 0;
      s[n] = s[m] + 1;
      for (int i = 0; i < numproj; i++)
        for (int j = 0; j < numproj; j++)
          for (int k = 0; k < numproj; k++)
            if (d[j][k][0])
              if (s[k] < s[j] + 1)
                s[k] = s[j] + 1;
    }
  {
    int h = s[0];
    for (int k = 0; k < numproj; k++)
      if (s[k] < h) h = s[k];
    for (int k = 0; k < numproj; k++)
      s[k] -= h;
  }
  Calculate();
}

void Delay()
{
  int n = -1, m;
  printf("Move starting point of ");
  while ((n <= 0) || (n > numproj))
    {
      scanf("%d", &n);
    }
  n--;
  printf("How far to the right? " );
  scanf("%d", &m);
  for (int k = 0; k < numproj; k++)
    {
      if (d[n][k][0])
        if (s[k] <= s[n] + 2 * m)
          {
            printf("Impossible because of %d.\n", k);
            return;
          }
      if (d[k][n][0])
        if (s[k] >= s[n] + 2 * m)
          {
            printf("Impossible because of %d.\n", k);
            return;
          }
    }
  s[n] = s[n] + 2 * m;
  {
    int h = s[0];
```

```
    for (int k = 0; k < numproj; k++)
      if (s[k] < h) h = s[k];
    for (int k = 0; k < numproj; k++)
      s[k] -= h;
  }
  Calculate();
}

void ChDimSeq()
{
  int n = -1, m = -1;
  printf("New dimension sequence for arrow from ");
  while ((n <= 0) || (n > numproj))
    {
      scanf("%d", &n);
    }
  printf("                                    to ");
  while ((m <= 0) || (m > numproj))
    {
      scanf("%d", &m);
    }
  n--; m--;
  if (!(d[n][m][0]))
    printf("There is no arrow from %d to %d.\n", n+1, m+1);
  else
    {
      printf("Dimension sequence (0 to finish)\n");
      for (int i = 0; i < MAXWIDTH; i++)
        {
          scanf("%d", &(d[n][m][i]));
          if (!d[n][m][i])
            {
              if (!i)
                {
                  d[n][m][0] = 1;
                  d[n][m][1] = 0;
                }
              i = MAXWIDTH;
            }
        }
    }
  Calculate();
}

void DelProj()
{
  if (numproj == 1)
    {
      printf("That would be the last one.\n");
      return;
    }
  int n = -1;
  printf("Delete tau-orbit ");
  while ((n <= 0) || (n > numproj))
    {
      scanf("%d", &n);
    }
  n--;
  int c = 0;
  for (int i = 0; i < numproj; i++)
    {
      if (d[i][n][0]) c++;
```

```
      if (d[n][i][0]) c++;
    }
  if (c > 1)
    {
      printf("That would destroy connectedness.\n");
      return;
    }
  numproj--;
  for (int i = n; i < numproj; i++)
    {
      s[i] = s[i+1];
      for (int k = 0; k <= numproj; k++)
        if (!(k == i))
          for (int j = 0; j < MAXWIDTH; j++)
            {
              d[i][k][j] = d[i+1][k][j];
              d[k][i][j] = d[k][i+1][j];
            }
    }
  {
    int h = s[0];
    for (int k = 0; k < numproj; k++)
      if (s[k] < h) h = s[k];
    for (int k = 0; k < numproj; k++)
      s[k] -= h;
  }
  Calculate();
}

void Calculate()
{
  // Calculate dimension vectors
  vector dvectors[MAXPROJ][MAXWIDTH];
  FILE* f; int w;
  for (int i = 0; i < numproj; i++)
    for (int j = 0; j < MAXWIDTH; j++)
      dvectors[i][j] = uvector(-1);
  for (int j = 0; j < MAXWIDTH; j++)
    for (int i = 0; i < numproj; i++)
      {
        if (j == s[i])
          {
            w = j;
            dvectors[i][j] = uvector(i);
            if (j)
              for (int k = 0; k < numproj; k++)
                dvectors[i][j] = dvectors[i][j] + d[k][i][0] * dvectors[k][j
                    -1];
          }
        if (j > s[i] + 1)
          if (!(dvectors[i][j-2] == uvector(-1)))
            {
              dvectors[i][j] = (-1) * dvectors[i][j-2];
              for (int k = 0; k < numproj; k++)
                {
                  int h = 0;
                  for (int l = s[k]; l < j; l++)
                    {
                      h++;
                      if (!(d[i][k][h])) h = 0;
                    }
```

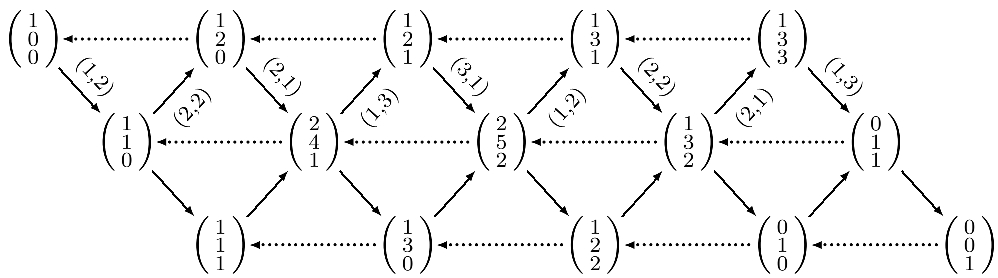
```
                dvectors[i][j] = dvectors[i][j] + d[i][k][h] * dvectors[k
                    ][j-1];
                h = 0;
                for (int l = s[i]; l < j; l++)
                  {
                    h++;
                    if (!(d[k][i][h])) h = 0;
                  }
                dvectors[i][j] = dvectors[i][j] + d[k][i][h] * dvectors[k
                    ][j-1];
              }
            if (!positive(dvectors[i][j]))
              dvectors[i][j] = uvector(-1);
            else
              w = j;
          }
      }
  // Write to file quiver.tex
  f = fopen("quiver.tex", "w");
  fprintf(f, BEGINDOC);
  for (int i = 0; i < numproj; i++)
    {
      for (int j = 0; j <= w; j++)
        {
          if (!(dvectors[i][j] == uvector(-1)))
            print(dvectors[i][j], f);
          if ((j > 0) && (j < w))
            if ((!(dvectors[i][j-1] == uvector(-1))) && (!(dvectors[i][j+1]
                == uvector(-1))))
              fprintf(f, " \\lDotsto ");
          fprintf(f, "&&");
        }
      fprintf(f, "\\\\\n&");
      if (i < numproj - 1)
        for (int j = 0; j < w; j++)
          {
            if (!(dvectors[i][j] == uvector(-1)))
              for (int k = i+1; k < numproj; k++)
                if (!(dvectors[k][j+1] == uvector(-1)))
                  {
                    if (d[i][k][0])
                      {
                        int h = 0;
                        for (int l = s[k]; l <= j; l++)
                          {
                            h++;
                            if (!(d[i][k][h])) h = 0;
                          }
                        if (d[i][k][h+1])
                          fprintf(f, "\\rdTo(2, %d)^{\\scriptstyle{(%d, %d)
                              }}", 2*(k-i), d[i][k][h], d[i][k][h+1]);
                        else
                          fprintf(f, "\\rdTo(2, %d)^{\\scriptstyle{(%d, %d)
                              }}", 2*(k-i), d[i][k][h], d[i][k][0]);
                      }
                    if (d[k][i][0])
                      {
                        int h = 0;
                        for (int l = s[i]; l <= j; l++)
                          {
                            h++;
                            if (!(d[k][i][h])) h = 0;
```

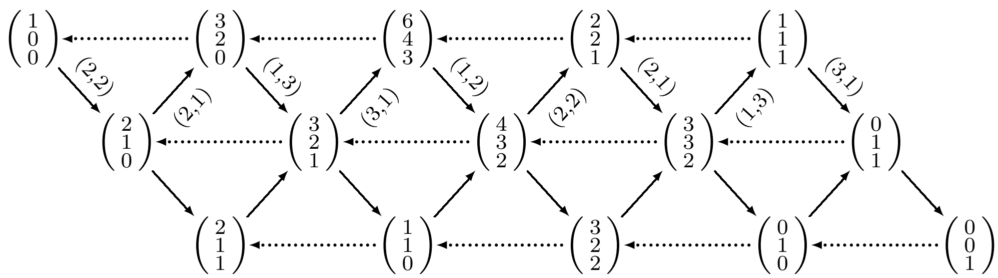
```
                          }
                        if (d[k][i][h+1])
                          fprintf(f, "\\rdTo(2,␣%d)^{\\scriptstyle{(%d,␣%d)
                              }}", 2*(k-i), d[k][i][h], d[k][i][h+1]);
                        else
                          fprintf(f, "\\rdTo(2,␣%d)^{\\scriptstyle{(%d,␣%d)
                              }}", 2*(k-i), d[k][i][h], d[k][i][0]);
                      }
                  }
            if (!(dvectors[i][j+1] == uvector(-1)))
              for (int k = i+1; k < numproj; k++)
                if (!(dvectors[k][j] == uvector(-1)))
                  {
                    if (d[i][k][0])
                      {
                        int h = 0;
                        for (int l = s[k]; l <= j; l++)
                          {
                            h++;
                            if (!(d[i][k][h])) h = 0;
                          }
                        if (d[i][k][h+1])
                          fprintf(f, "\\ruTo(2,␣%d)_{\\scriptstyle{(%d,␣%d)
                              }}", 2*(k-i), d[i][k][h], d[i][k][h+1]);
                        else
                          fprintf(f, "\\ruTo(2,␣%d)_{\\scriptstyle{(%d,␣%d)
                              }}", 2*(k-i), d[i][k][h], d[i][k][0]);
                      }
                    if (d[k][i][0])
                      {
                        int h = 0;
                        for (int l = s[i]; l <= j; l++)
                          {
                            h++;
                            if (!(d[k][i][h])) h = 0;
                          }
                        if (d[k][i][h+1])
                          fprintf(f, "\\ruTo(2,␣%d)_{\\scriptstyle{(%d,␣%d)
                              }}", 2*(k-i), d[k][i][h], d[k][i][h+1]);
                        else
                          fprintf(f, "\\ruTo(2,␣%d)_{\\scriptstyle{(%d,␣%d)
                              }}", 2*(k-i), d[k][i][h], d[k][i][0]);
                    }
                }
          fprintf(f, "&&");
        }
      fprintf(f, "\\\\\n");
    }
  fprintf(f, ENDDOC);
  fclose(f);
  system(TEXCOMMAND);
  system(PSCOMMAND);
}

vector operator +(vector v1, vector v2)
{
  vector v;
  for (int i = 0; i < MAXPROJ; i++) v.e[i] = v1.e[i] + v2.e[i];
  return v;
}

vector operator *(int x, vector v)
```

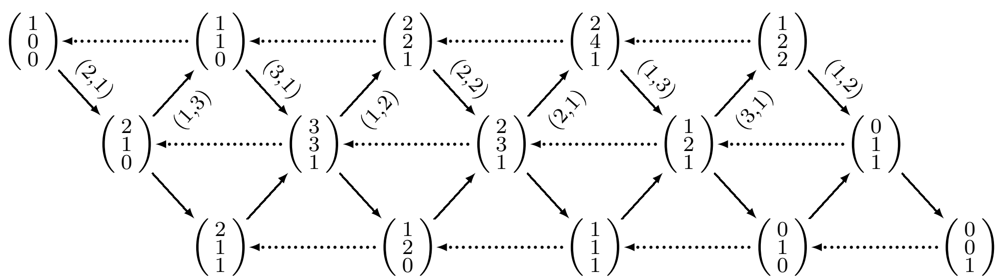
```
{
  for (int i = 0; i < MAXPROJ; i++) v.e[i] *= x;
  return v;
}

vector uvector(int i) // gives the i-th unit vector
{
  vector v;
  for (int j = 0; j < MAXPROJ; j++) v.e[j] = i==j?1:0;
  return v;
}

int operator ==(vector v1, vector v2)
{
  int g = 1;
  for (int i = 0; i < MAXPROJ; i++)
    if (v1.e[i] != v2.e[i]) g = 0;
  return g;
}

void print(vector v, FILE* f)
{
  fprintf(f, "␣\\left(␣\\begin{smallmatrix}␣%d", v.e[0]);
  for (int i = 1; i < numproj; i++) fprintf(f, "\\\\␣%d", v.e[i]);
  fprintf(f, "␣\\end{smallmatrix}␣\\right)␣");
}

int positive(vector v)
{
 int g = 1;
  for (int i = 0; i < MAXPROJ; i++)
    if (v.e[i] < 0) g = 0;
  return g;
}
```

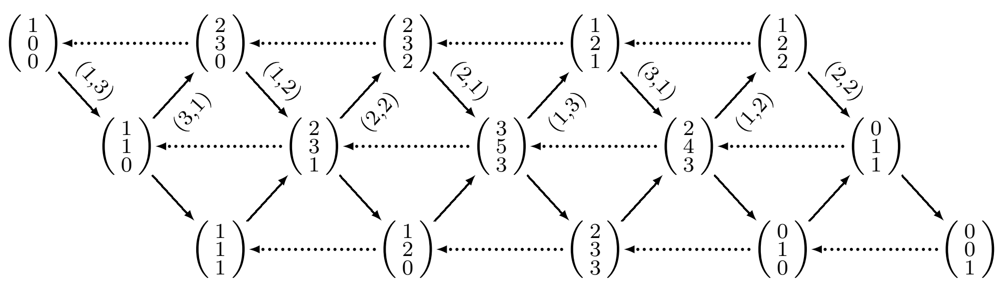
C Auslander-Reiten quivers of hereditary artinian rings of type H_3 and H_4

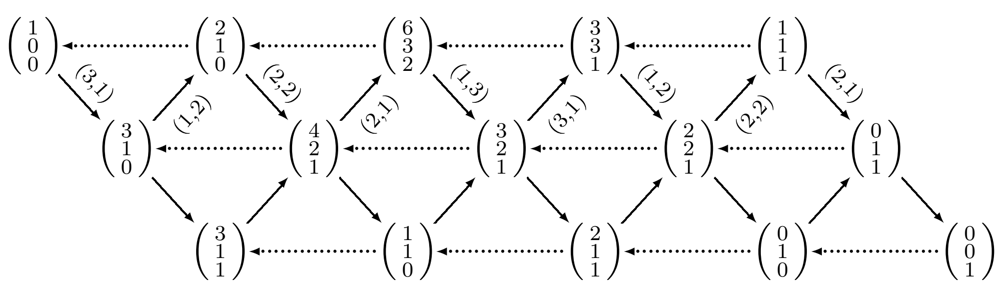
Translation quivers of type H_3 with the cyclic permutations of the dimension sequence:



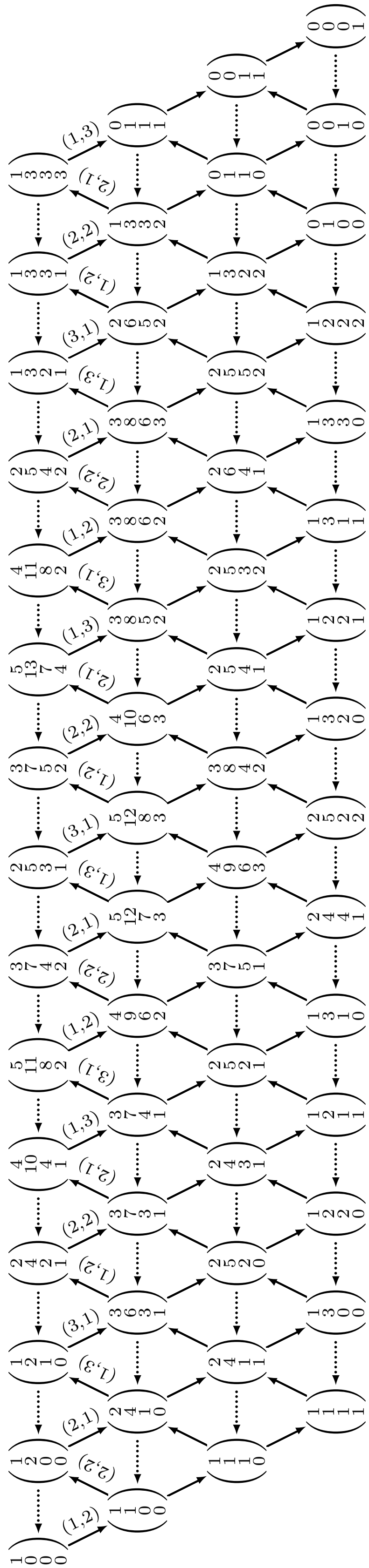


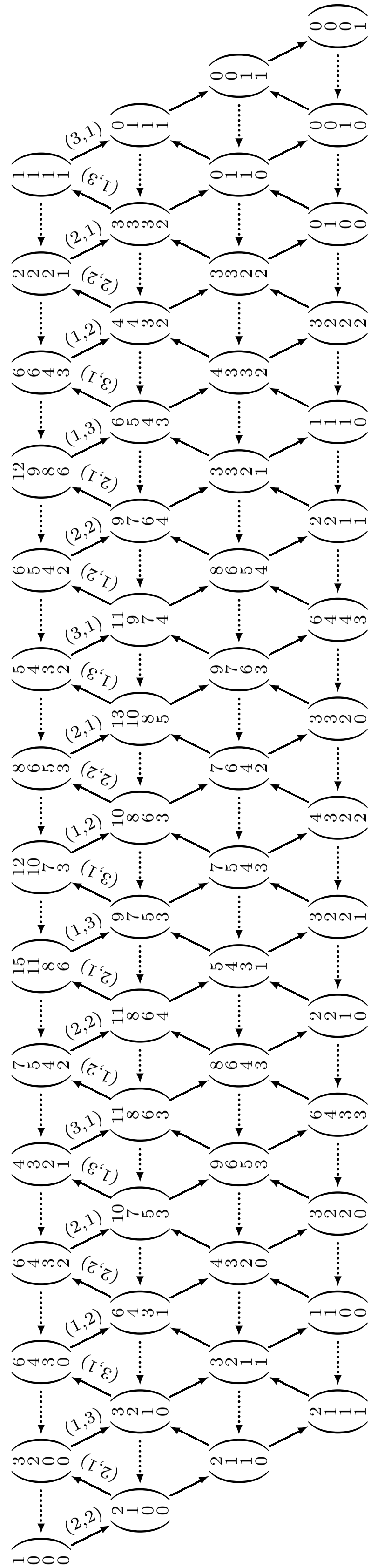


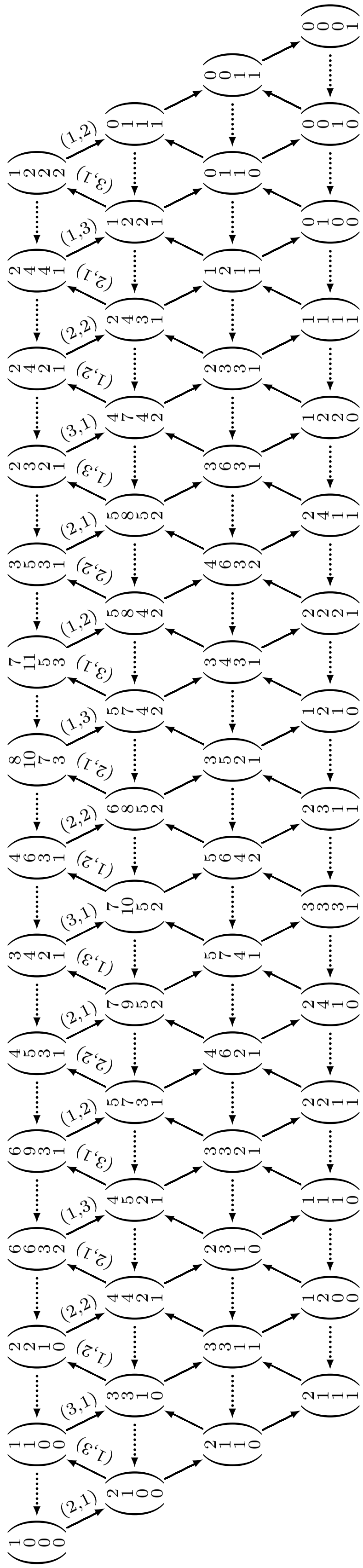


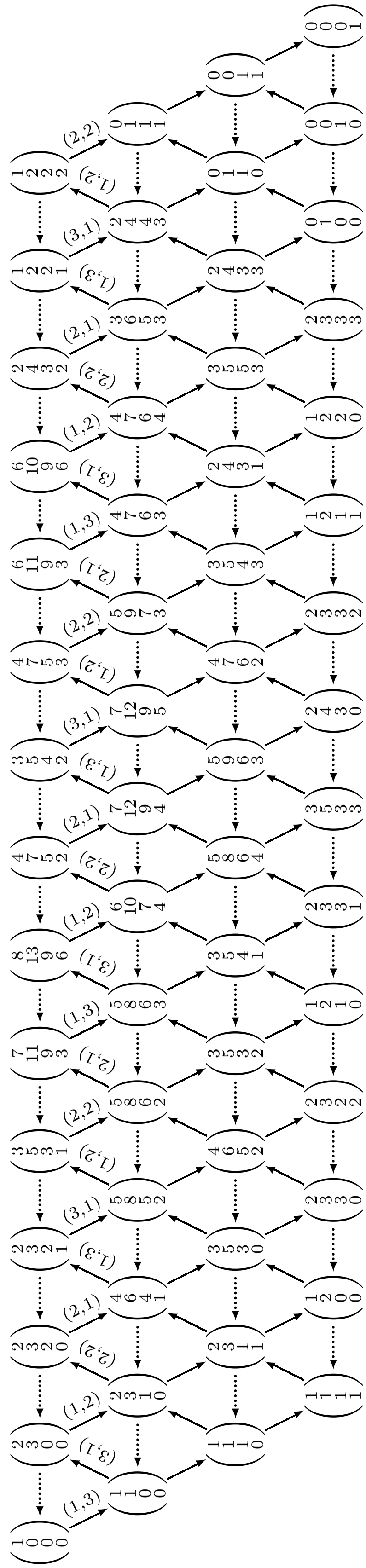


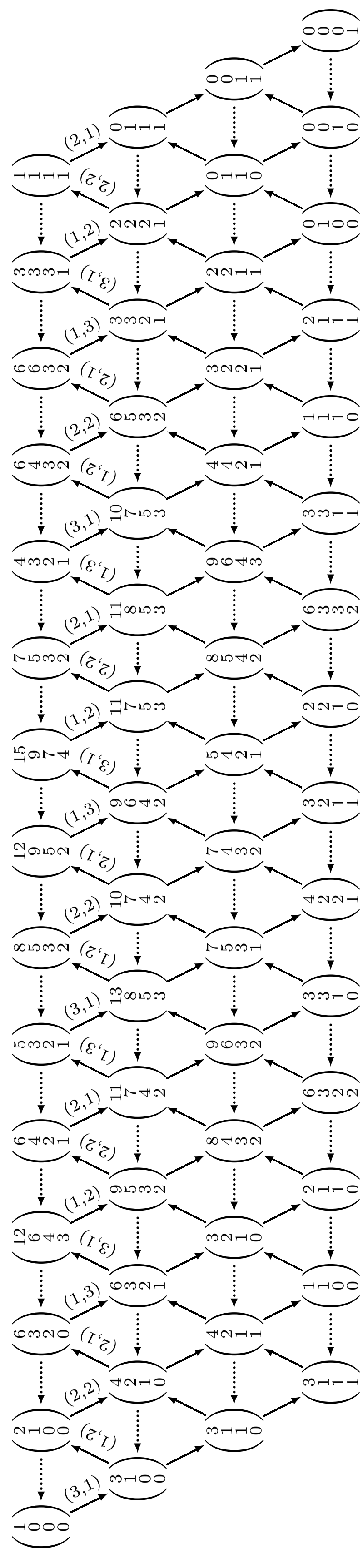
Translation quivers of type H_4 with the cyclic permutations of the dimension sequence:











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