THE DIMENSION OF THE DERIVED CATEGORY OF ELLIPTIC CURVES AND TUBULAR WEIGHTED PROJECTIVE LINES

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Abstract. We show that the dimension of the derived category of an elliptic curve or a tubular weighted projective line is one. We give explicit generators realizing this number, and show that they are in a certain sense minimal.

1. Introduction

The dimension of a triangulated category has been introduced by Rouquier in [20]. It measures how long it takes to build the entire triangulated category starting from just one object. In subsequent papers [3, 4, 5, 6, 7, 11, 15, 16, 19] this has been shown to be a useful invariant.

Rouquier has shown in [20, Proposition 7.4] that the dimension of the derived category of an algebra is bounded above by the global dimension. However this is not true for abelian categories in general (see Remark 4.2). In this paper, two tubular situations are studied: coherent sheaves over an elliptic curve, and coherent sheaves over a weighted projective line of tubular type. We show that the dimension of the derived category is one in both cases. Orlov has independently shown that the dimension of the derived category of a smooth projective curve is exactly one [17]. This implies our result in the case of elliptic curves. Orlov's methods are based on geometric information. Our methods simultaneously cover elliptic curves and tubular weighted projective lines, and are based on explicit knowledge of the category of coherent sheaves. The following result describing exactly what kind of sheaves generate the derived category in one step is our main result of this paper (for the notation see Section 2):

Theorem. *Let $\mathbb{X}$ be an elliptic curve or a weighted projective line of tubular type.*

(1) *Let $\mathcal{F}$ be any sheaf, such that the indecomposable direct summands of $\mathcal{F}$ have at most two different slopes. Then $\langle \mathcal{F} \rangle_2 \neq D^b(\operatorname{coh} \mathbb{X})$.*

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Here $\langle \mathcal{F} \rangle_2$ *denotes the full subcategory of all objects which are direct summands of cones of maps between objects in* $\operatorname{add}\{\mathcal{F}[i] \mid i \in \mathbb{Z}\}$ *(see Definition 2.6).*

(2) *Given any three tubes of pairwise different slope, there is a sheaf* $\mathcal{F}$ *having indecomposable direct summands only in these three tubes, such that* $\langle \mathcal{F} \rangle_2 = D^b(\operatorname{coh} \mathbb{X})$.

The main ideas for the proof are the following:

We prove (1) by showing that for such a sheaf $\mathcal{F}$, any indecomposable sheaf $\mathcal{G}$ which is of the same slope as some indecomposable direct summands of $\mathcal{F}$, but does not lie in the same tube as any summand of $\mathcal{F}$, is not contained in $\langle \mathcal{F} \rangle_2$.

For the proof of (2) we first note that since $\operatorname{coh} \mathbb{X}$ is hereditary, it suffices to show that $\operatorname{coh} \mathbb{X} \subset \langle \mathcal{F} \rangle_2$. We divide $\operatorname{coh} \mathbb{X}$ into the three parts $\operatorname{coh}_{\leq \frac{\mu_1+\mu_2}{2}} \mathbb{X}$, $\operatorname{coh}_{> \frac{\mu_2+\mu_3}{2}} \mathbb{X}$, and $\operatorname{coh}_{> \frac{\mu_1+\mu_2}{2}} \mathbb{X} \cap \operatorname{coh}_{\leq \frac{\mu_2+\mu_3}{2}}$ according to the slopes of the sheaves, where $\mu_1 < \mu_2 < \mu_3$ are the slopes of the tubes of the theorem. We show that one can resolve all sheaves in $\operatorname{coh}_{> \frac{\mu_2+\mu_3}{2}} \mathbb{X}$ by sheaves in the first two tubes, and similarly for sheaves in $\operatorname{coh}_{\leq \frac{\mu_1+\mu_2}{2}} \mathbb{X}$. Finally all sheaves in $\operatorname{coh}_{> \frac{\mu_1+\mu_2}{2}} \mathbb{X} \cap \operatorname{coh}_{\leq \frac{\mu_2+\mu_3}{2}}$ are obtained as images of a map from a sheaf in the first to a sheaf in the last tube. In each case it follows that the sheaves are in $\langle \mathcal{F} \rangle_2$, for a suitable $\mathcal{F}$.

In Section 2 we first introduce the notation and recall some well-known facts about the categories of coherent sheaves over elliptic curves and weighted projective lines of tubular type. Then we give Rouquier's definition of the dimension of a triangulated category, and some immediate consequences of that definition.

In the third section we study what part of the category of coherent sheaves is generated by certain sheaves. We explicitly calculate rank and Euler characteristic of the minimal right- and left approximations, and therefore also obtain these values for the kernels.

In Section 4 we apply the results of Section 3 to obtain our main theorem.

2. Notation and background

2.1. **Elliptic curves and tubular weighted projective lines.** Throughout this paper we assume k to be an algebraically closed field. Moreover, we assume that we have one of the following setups:

(1) $\mathbb{X}$ is an elliptic curve, or
(2) $\mathbb{X}$ is a weighted projective line of tubular type, that is, of type $(2, 2, 2, 2)$, $(3, 3, 3)$, $(2, 4, 4)$ or $(2, 3, 6)$ – see [10], in particular [10,

5.4.2] (also see [9] for a study of general sheaves, somewhat related to what we do in this paper).

2.1. **Remarks.** (1) By [10, Section 4], the category $\operatorname{coh}\mathbb{X}$ of coherent sheaves over a weighted projective line $\mathbb{X}$ is derived equivalent to the category of modules over a canonical algebra. Hence all results on the derived categories of tubular weighted projective lines are automatically also results on the derived categories of tubular algebras (see [18, Chapter 5] – also see [21] for an introduction to tubular algebras).

(2) By [10, 5.4.1], weighted projective lines of "smaller" type (that is of Euler characteristic $\overline{\chi}(\mathcal{O}) > 0$) are derived equivalent to tame hereditary algebras. Hence the dimension of their derived category is also 1.

We study the category $\operatorname{coh}\mathbb{X}$ of coherent sheaves on $\mathbb{X}$. The following Facts 2.2 - 2.5 are well-known (see [1, 8], in particular [8, Summary below 4.26], for the case of elliptic curves and [10, 13], in particular [10, Theorem 5.6], for the case of tubular weighted projective lines; see also [12] for a good summary of the basics in both cases).

2.2. **Fact.** *The category* $\operatorname{coh}\mathbb{X}$ *is a hereditary,* Hom-*finite abelian category, which has AR-sequences. Moreover all AR-components are tubes, so in particular the AR-translation τ has finite order.*

For $\mathcal{F} \in \operatorname{coh}\mathbb{X}$ we denote by $p(\mathcal{F})$ the minimal positive integer such that $\tau^{p(\mathcal{F})}\mathcal{F} = \mathcal{F}$. Moreover we set $\overline{\mathcal{F}} = \oplus_{i=1}^{p(\mathcal{F})}\tau^i\mathcal{F}$.

2.3. **Definition.** Let $\mathcal{F} \in \operatorname{coh}\mathbb{X}$. We denote by $\operatorname{rk}\mathcal{F}$ the rank of $\mathcal{F}$, and by $\chi(\mathcal{F}) = \dim_k \operatorname{Hom}(\mathcal{O},\mathcal{F}) - \dim_k \operatorname{Ext}(\mathcal{O},\mathcal{F})$ its Euler characteristic. Moreover, we set $\overline{\chi}(\mathcal{F}) = \frac{1}{p(\mathcal{F})}\chi(\overline{\mathcal{F}})$. That is $\overline{\chi}(\mathcal{F})$ is the average Euler characteristic over the τ-translates of $\mathcal{F}$. Finally we denote by $\mu(\mathcal{F}) = \overline{\chi}(\mathcal{F})/\operatorname{rk}(\mathcal{F})$ the slope of $\mathcal{F}$ (we set $\mu(\mathcal{F}) = \infty$ if $\operatorname{rk}(\mathcal{F}) = 0$, that is if $\mathcal{F}$ is a torsion sheaf).

Note that rk, χ and $\overline{\chi}$ are additive on short exact sequences. Hence they can be defined as maps on the Grothendieck group.

We denote by $\operatorname{ind}\mathbb{X}$ a fixed set of representatives of isomorphism classes of indecomposable sheaves, by $\operatorname{ind}_\mu\mathbb{X} = \{\mathcal{F} \in \operatorname{ind}\mathbb{X} \mid \mu(\mathcal{F}) = \mu\}$, by $\operatorname{ind}_{>\mu}\mathbb{X} = \{\mathcal{F} \in \operatorname{ind}\mathbb{X} \mid \mu(\mathcal{F}) > \mu\}$, and similar variations.

Moreover we set $\operatorname{coh}_\mu\mathbb{X} = \operatorname{add}\operatorname{ind}_\mu\mathbb{X}$. This is the category of semi-stable sheaves of slope μ, which is (except for $\mu = \infty$) strictly smaller than the category of all sheaves of slope μ.

2.4. **Fact.** *All sheaves in one AR-component have the same slope. For a fixed slope μ the category* $\operatorname{ind}_\mu \mathbb{X}$ *decomposes into a one-parameter family of uniserial categories.*

In case $\mathbb{X}$ is an elliptic curve this family is indexed by points on the curve, and all the uniserial categories are tubes of rank 1*.*

In case $\mathbb{X}$ is a weighted projective line of type $(p_0, \dots, p_n)$ the family is indexed by points in $\mathbb{P}^1_k$, all but finitely many of the uniserial categories are tubes of rank 1*, and the remaining ones are tubes of the rank $p_0, \dots, p_n$.*

The prefix "*quasi-*" indicates that some concept is applied to the abelian category of objects in one specific tube. For instance the objects in the mouth of a tube are called quasi-simple. We denote by q.Rad, q.Soc, and q.length the quasi-radical, quasi-socle, and quasi-length, respectively.

Note that $\overline{\mathcal{F}}$ and $\mathcal{F}$, and hence also $\chi(\mathcal{F})$ and $\overline{\chi}(\mathcal{F})$ coincide whenever $\mathcal{F}$ is in a homogeneous tube. In particular they always coincide if $\mathbb{X}$ is an elliptic curve.

For $\mathcal{F}, \mathcal{G} \in \operatorname{coh} \mathbb{X}$, we set

$$\langle \mathcal{F}, \mathcal{G} \rangle = \dim_k \operatorname{Hom}(\mathcal{F}, \mathcal{G}) - \dim_k \operatorname{Ext}(\mathcal{F}, \mathcal{G}), \text{ and}$$
$$\langle\!\langle \mathcal{F}, \mathcal{G} \rangle\!\rangle = \frac{1}{p(\mathcal{G})} \left\langle \mathcal{F}, \overline{\mathcal{G}} \right\rangle.$$

As for χ, we note that these notions coincide whenever $\mathcal{F}$ or $\mathcal{G}$ lies in a homogeneous tube, so in particular they coincide for elliptic curves.

2.5. **Fact.** (1) *For $\mu > \nu$ we have* $\operatorname{Hom}(\operatorname{coh}_\mu \mathbb{X}, \operatorname{coh}_\nu \mathbb{X}) = 0$*, and we have* $\operatorname{Ext}(\operatorname{coh}_\nu \mathbb{X}, \operatorname{coh}_\mu \mathbb{X}) = 0$*.*

(2) *(Riemann-Roch theorem – see* [8, Theorem 4.13] *for elliptic curves and* [10, 2.9] *and* [12, page 25] *for weighted projective lines) For any $\mathcal{F}, \mathcal{G} \in \operatorname{coh} \mathbb{X}$ we have*

$$\langle\!\langle \mathcal{F}, \mathcal{G} \rangle\!\rangle = \operatorname{rk}(\mathcal{F}) \overline{\chi}(\mathcal{G}) - \overline{\chi}(\mathcal{F}) \operatorname{rk}(\mathcal{G})$$

2.2. **Rouquier's dimension of a triangulated category.** Let $\mathbf{T}$ be a triangulated category. We recall Rouquier's definition (see [20]) of the dimension $\dim \mathbf{T}$.

2.6. **Definition.** Let $X \in \mathcal{Ob}\, \mathbf{T}$. We set

$$\begin{aligned} \langle X \rangle_0 &= 0, \\ \langle X \rangle_1 &= \operatorname{add}\{X[i] \mid i \in \mathbb{Z}\}, \\ \langle X \rangle_{n+1} &= \operatorname{add}\{\operatorname{Cone}(f) \mid f \in \operatorname{Hom}(\langle X \rangle_n, \langle X \rangle_1)\}, \text{ and} \\ \langle X \rangle_\infty &= \cup_{n \in \mathbb{N}} \langle X \rangle_n . \end{aligned}$$

The *dimension of* $\mathbf{T}$ is

$$\dim \mathbf{T} = \inf\{n \in \mathbb{N}_0 \mid \exists X \text{ such that } \langle X \rangle_{n+1} = \mathbf{T}\}.$$

2.7. **Observation.** *If* $\mathbf{T}$ *is not finite up to shifts (in particular in both situations we study here) we have* $\dim \mathbf{T} \geq 1$.

2.8. **Observation.** *Let* $\mathbf{T}$ *be a triangulated category with* $\dim \mathbf{T} < \infty$. *Let* $X \in \mathcal{O}b\,\mathbf{T}$ *such that* $\langle X \rangle_\infty = \mathbf{T}$. *Then there is some* $n \in \mathbb{N}$ *such that* $\langle X \rangle_n = \mathbf{T}$.

Proof. Since $\dim \mathbf{T} < \infty$, there is $Y \in \mathcal{O}b\,\mathbf{T}$ and $n \in \mathbb{N}$ such that $\langle Y \rangle_n = \mathbf{T}$. Since $Y \in \langle X \rangle_\infty$, we have $Y \in \langle X \rangle_m$, for some $m \in \mathbb{N}$. Hence $\langle X \rangle_{mn} = \langle Y \rangle_n = \mathbf{T}$. □

3. Approximations

In this section we determine the class of coherent sheaves generated and cogenerated by a fixed sheaf $\mathcal{F}$ with the following properties:

(1) all direct summands of $\mathcal{F}$ have the same slope $\neq \infty$, and
(2) $\tau\mathcal{F} \in \operatorname{add}\mathcal{F}$.

We denote by $\mathcal{G}en\,\mathcal{F}$ the category of all sheaves generated by $\mathcal{F}$, that is, the category of all sheaves $\mathcal{G}$ such that there is an epimorphism $\mathcal{F}^n \twoheadrightarrow \mathcal{G}$ for some n. Dually we denote by $\mathcal{C}og\,\mathcal{F}$ the category of sheaves cogenerated by $\mathcal{F}$, that is, the category of all sheaves $\mathcal{G}$ such that there is a monomorphism $\mathcal{G} \rightarrowtail \mathcal{F}^n$ for some n.

3.1. **Remark.** If the slope of all direct summands of $\mathcal{F}$ is ∞ (that is, if $\mathcal{F}$ is a torsion sheaf), then $\mathcal{F}$ does not generate or cogenerate any sheaves of other slopes. We restrict to the case of finite slope to avoid having two cases in all the following results.

Let I be the set of tubes containing an indecomposable direct summand of $\mathcal{F}$. For each $\mathbf{t} \in I$, we choose some indecomposable $\mathcal{F}_\mathbf{t}$, which is of maximal quasi-length in $\operatorname{add}\mathcal{F} \cap \mathbf{t}$. Then

$$\operatorname{add}\{\tau^i\mathcal{F}_\mathbf{t} \mid i \in \mathbb{Z}, \mathbf{t} \in I\} \subseteq \operatorname{add}\mathcal{F} \subseteq \operatorname{add}\{\text{q.Rad}^j\,\tau^i\mathcal{F}_\mathbf{t} \mid i,j \in \mathbb{Z}, j \geq 0, \mathbf{t} \in I\}$$

and therefore

$$\mathcal{G}en\,\mathcal{F} = \mathcal{G}en\{\tau^i\mathcal{F}_\mathbf{t} \mid i \in \mathbb{Z}, \mathbf{t} \in I\} \text{ and}$$
$$\mathcal{C}og\,\mathcal{F} = \mathcal{C}og\{\tau^i\mathcal{F}_\mathbf{t} \mid i \in \mathbb{Z}, \mathbf{t} \in I\}.$$

With the notation above we set

$$\delta(\mathcal{F}) = \frac{1}{\sum_{\mathbf{t}\in I} \frac{(\operatorname{rk}\mathcal{F}_\mathbf{t})^2 p(\mathcal{F}_\mathbf{t})}{\text{q.length}\,\mathcal{F}_\mathbf{t}}}$$

3.2. **Definition.** Let $\mathcal{G} \in \operatorname{coh}\mathbb{X}$. A *right* $\mathcal{F}$*-approximation of* $\mathcal{G}$ is a map $f\colon \mathcal{F}' \longrightarrow \mathcal{G}$, with $\mathcal{F}' \in \operatorname{add}\mathcal{F}$, such that any other map from $\operatorname{add}\mathcal{F}$ to $\mathcal{G}$ factors through f. Equivalently one could require that the map $\operatorname{Hom}(\mathcal{F},\mathcal{F}') \xrightarrow{f_*} \operatorname{Hom}(\mathcal{F},\mathcal{G})$ is onto.

A *minimal right $\mathcal{F}$-approximation of $\mathcal{G}$* is a right $\mathcal{F}$-approximation, which is right minimal in the sense of [2, I.2], that is, it does not vanish on any direct summand of $\mathcal{F}'$.

Left approximations, and minimal left approximations, are defined dually.

3.3. **Proposition.** (1) *Let $\mathcal{G} \in \operatorname{coh}_{>\mu(\mathcal{F})} \mathbb{X}$, and assume $\mathcal{F}' \longrightarrow \mathcal{G}$ is a minimal right $\mathcal{F}$-approximation. Then*

$$\operatorname{rk}(\mathcal{F}') = \frac{\overline{\chi}(\mathcal{G}) - \operatorname{rk}(\mathcal{G})\mu(\mathcal{F})}{\delta(\mathcal{F})} \quad \text{and}$$
$$\overline{\chi}(\mathcal{F}') = \frac{\overline{\chi}(\mathcal{G})\mu(\mathcal{F}) - \operatorname{rk}(\mathcal{G})\mu(\mathcal{F})^2}{\delta(\mathcal{F})}.$$

(2) *Let $\mathcal{G} \in \operatorname{coh}_{<\mu(\mathcal{F})} \mathbb{X}$, and assume $\mathcal{G} \longrightarrow \mathcal{F}''$ is a minimal left $\mathcal{F}$-approximation. Then*

$$\operatorname{rk}(\mathcal{F}'') = \frac{\operatorname{rk}(\mathcal{G})\mu(\mathcal{F}) - \overline{\chi}(\mathcal{G})}{\delta(\mathcal{F})} \quad \text{and}$$
$$\overline{\chi}(\mathcal{F}'') = \frac{\operatorname{rk}(\mathcal{G})\mu(\mathcal{F})^2 - \overline{\chi}(\mathcal{G})\mu(\mathcal{F})}{\delta(\mathcal{F})}.$$

Proof. We only prove (1), the proof of (2) is dual. Moreover it suffices to prove the formula for $\operatorname{rk} \mathcal{F}'$, the formula for $\overline{\chi}(\mathcal{F}')$ then follows from $\mu(\mathcal{F}) = \mu(\mathcal{F}') = \frac{\overline{\chi}(\mathcal{F}')}{\operatorname{rk}(\mathcal{F}')}$.

Any map from a sheaf in the intersection of $\operatorname{add} \mathcal{F}$ with $\mathbf{t}$ to $\mathcal{G}$ factors through some τ-shift of $\mathcal{F}_{\mathbf{t}}$. Therefore the minimal right $\mathcal{F}$-approximation of $\mathcal{G}$ is of the form

$$f \colon \mathcal{F}' = \oplus_{\mathbf{t} \in I} \oplus_{i=1}^{p(\mathcal{F}_{\mathbf{t}})} (\tau^i \mathcal{F}_{\mathbf{t}})^{d_{\mathbf{t}i}} \longrightarrow \mathcal{G}.$$

The short exact sequence $\operatorname{q.Soc} \tau^i \mathcal{F}_{\mathbf{t}} \hookrightarrow \tau^i \mathcal{F}_{\mathbf{t}} \twoheadrightarrow \tau^i \mathcal{F}_{\mathbf{t}} / \operatorname{q.Soc} \tau^i \mathcal{F}_{\mathbf{t}}$ induces an exact sequence

$$\operatorname{Hom}(\frac{\tau^i \mathcal{F}_{\mathbf{t}}}{\operatorname{q.Soc} \tau^i \mathcal{F}_{\mathbf{t}}}, \mathcal{G}) \rightarrowtail \operatorname{Hom}(\tau^i \mathcal{F}_{\mathbf{t}}, \mathcal{G}) \twoheadrightarrow \operatorname{Hom}(\operatorname{q.Soc} \tau^i \mathcal{F}_{\mathbf{t}}, \mathcal{G}).$$

Since any map $\frac{\tau^i \mathcal{F}_{\mathbf{t}}}{\operatorname{q.Soc} \tau^i \mathcal{F}_{\mathbf{t}}} \longrightarrow \mathcal{G}$ factors through $\tau^{i-1} \mathcal{F}_{\mathbf{t}}$ these maps do not occur in the minimal right approximation, and hence

$$d_{\mathbf{t}i} = \dim_k \operatorname{Hom}(\operatorname{q.Soc} \tau^i \mathcal{F}_{\mathbf{t}}, \mathcal{G}).$$

Now we can directly calculate the rank of $\mathcal{F}'$:

$$\begin{aligned}
\operatorname{rk}\mathcal{F}' &= \operatorname{rk}(\oplus_{\mathbf{t}} \oplus_{i=1}^{p(\mathcal{F}_{\mathbf{t}})} (\tau^i \mathcal{F}_{\mathbf{t}})^{d_{\mathbf{t}i}}) \\
&= \sum_{\mathbf{t}\in I} \sum_{i=1}^{p(\mathcal{F}_{\mathbf{t}})} \operatorname{rk}(\tau^i \mathcal{F}_{\mathbf{t}}) \dim_k \operatorname{Hom}(\text{q.Soc}\,\tau^i \mathcal{F}_{\mathbf{t}}, \mathcal{G}) \\
&= \sum_{\mathbf{t}\in I} \operatorname{rk}(\mathcal{F}_{\mathbf{t}}) \sum_{i=1}^{p(\mathcal{F}_{\mathbf{t}})} \dim_k \operatorname{Hom}(\text{q.Soc}\,\tau^i \mathcal{F}_{\mathbf{t}}, \mathcal{G}) \\
&= \sum_{\mathbf{t}\in I} \operatorname{rk}(\mathcal{F}_{\mathbf{t}}) p(\mathcal{F}_{\mathbf{t}}) \langle\!\langle \text{q.Soc}\,\mathcal{F}_{\mathbf{t}}, \mathcal{G} \rangle\!\rangle \\
&= \sum_{\mathbf{t}\in I} \operatorname{rk}(\mathcal{F}_{\mathbf{t}}) p(\mathcal{F}_{\mathbf{t}}) [\operatorname{rk}(\text{q.Soc}\,\mathcal{F}_{\mathbf{t}}) \overline{\chi}(\mathcal{G}) - \overline{\chi}(\text{q.Soc}\,\mathcal{F}_{\mathbf{t}}) \operatorname{rk}(\mathcal{G})] \\
&= \sum_{\mathbf{t}\in I} \frac{(\operatorname{rk}\mathcal{F}_{\mathbf{t}})^2 p(\mathcal{F}_{\mathbf{t}})}{\text{q.length}\,\mathcal{F}_{\mathbf{t}}} [\overline{\chi}(\mathcal{G}) - \mu(\mathcal{F}_{\mathbf{t}}) \operatorname{rk}(\mathcal{G})] \\
&= (\overline{\chi}(\mathcal{G}) - \mu(\mathcal{F}) \operatorname{rk}(\mathcal{G})) \sum_{\mathbf{t}\in I} \frac{(\operatorname{rk}\mathcal{F}_{\mathbf{t}})^2 p(\mathcal{F}_{\mathbf{t}})}{\text{q.length}\,\mathcal{F}_{\mathbf{t}}}
\end{aligned}$$

The claim follows by definition of $\delta(\mathcal{F})$. □

The following observation follows immediately from the fact that any sheaf has non-negative rank, and moreover that any sheaf of rank 0 has slope ∞, and that therefore, by 2.5, there are no morphisms from it to sheaves of any other slope.

3.4. **Observation.** *Let* $\mathcal{G} \xrightarrow{f} \mathcal{G}' \in \operatorname{coh}\mathbb{X}$. *Then*

$$\begin{aligned}
f \textit{ mono } &\Rightarrow \operatorname{rk}\mathcal{G} \leq \operatorname{rk}\mathcal{G}', \\
f \textit{ epi } &\Rightarrow \operatorname{rk}\mathcal{G} \geq \operatorname{rk}\mathcal{G}'.
\end{aligned}$$

If, in addition, $\mathcal{G} \in \operatorname{coh}_{<\infty}\mathbb{X}$ *and* f *is not an isomorphism then*

$$f \textit{ epi } \Rightarrow \operatorname{rk}\mathcal{G} > \operatorname{rk}\mathcal{G}'.$$

3.5. **Corollary.** (1) *Let* $\mathcal{G} \in \operatorname{ind}_{>\mu(\mathcal{F})}\mathbb{X}$ *and let* $f\colon \mathcal{F}' \longrightarrow \mathcal{G}$ *be a minimal right* $\mathcal{F}$*-approximation. Then*

$$\begin{aligned}
f \textit{ mono } &\Rightarrow \mu(\mathcal{G}) \leq \mu(\mathcal{F}) + \delta(\mathcal{F}), \\
f \textit{ epi } &\Rightarrow \mu(\mathcal{G}) > \mu(\mathcal{F}) + \delta(\mathcal{F}).
\end{aligned}$$

(2) *Let* $\mathcal{G} \in \operatorname{ind}_{<\mu(\mathcal{F})}\mathbb{X}$ *and let* $f\colon \mathcal{G} \longrightarrow \mathcal{F}'$ *be a minimal left* $\mathcal{F}$*-approximation. Then*

$$\begin{aligned}
f \textit{ mono } &\Rightarrow \mu(\mathcal{G}) \leq \mu(\mathcal{F}) - \delta(\mathcal{F}), \\
f \textit{ epi } &\Rightarrow \mu(\mathcal{G}) > \mu(\mathcal{F}) - \delta(\mathcal{F}).
\end{aligned}$$

We now show that the implications in Corollary 3.5 are actually equivalences.

3.6. **Theorem.** (1) *Let* $\mathcal{G} \in \operatorname{ind}_{>\mu(\mathcal{F})} \mathbb{X}$ *and let* $f \colon \mathcal{F}' \longrightarrow \mathcal{G}$ *be a minimal right* $\mathcal{F}$*-approximation. Then*

$$f \text{ mono} \iff \mu(\mathcal{G}) \leq \mu(\mathcal{F}) + \delta(\mathcal{F}),$$
$$f \text{ epi} \iff \mu(\mathcal{G}) > \mu(\mathcal{F}) + \delta(\mathcal{F}).$$

In particular,

$$\mathcal{G}en\,\mathcal{F} = \operatorname{add}(\{\text{q.Rad}^j\, \tau^i \mathcal{F}_{\mathbf{t}} \mid i, j \in \mathbb{N}, \mathbf{t} \in I\} \cup \operatorname{ind}_{>\mu(\mathcal{F})+\delta(\mathcal{F})} \mathbb{X}).$$

(2) *Let* $\mathcal{G} \in \operatorname{ind}_{<\mu(\mathcal{F})} \mathbb{X}$ *and let* $f \colon \mathcal{G} \longrightarrow \mathcal{F}'$ *be a minimal left* $\mathcal{F}$*-approximation. Then*

$$f \text{ mono} \iff \mu(\mathcal{G}) \leq \mu(\mathcal{F}) - \delta(\mathcal{F}),$$
$$f \text{ epi} \iff \mu(\mathcal{G}) > \mu(\mathcal{F}) - \delta(\mathcal{F}).$$

In particular,

$$\mathcal{C}og\,\mathcal{F} = \operatorname{add}(\{\text{q.Rad}^j\, \tau^i \mathcal{F}_{\mathbf{t}} \mid i, j \in \mathbb{N}, \mathbf{t} \in I\} \cup \operatorname{ind}_{\leq\mu(\mathcal{F})-\delta(\mathcal{F})} \mathbb{X}).$$

Proof. We only prove the first part, the proof for the second one is similar. Let $\mathcal{G}$, $\mathcal{F}'$, and f be as in part (1) of the theorem, and let $\mathcal{H}$ be the image of f. Since the induced map $\mathcal{F}' \longrightarrow \mathcal{H}$ is epi, by Corollary 3.5 we have $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}''$ with $\mathcal{H}' \in \operatorname{coh}_{\mu(\mathcal{F})} \mathbb{X}$ and $\mathcal{H}'' \in \operatorname{coh}_{>\mu(\mathcal{F})+\delta(\mathcal{F})} \mathbb{X}$. It is easily seen that $\mathcal{H}' \in \operatorname{add} \mathcal{F}$. We denote the cokernel of the induced map $\mathcal{H}'' \hookrightarrow \mathcal{G}$ by $\mathcal{G}'$. Then we have the left diagram below, and applying $\operatorname{Hom}(\mathcal{F}, -)$ to it we obtain the right diagram.

$$\begin{array}{ccc} \mathcal{H}'' & = & \mathcal{H}'' \\ {\scriptstyle (0,1)}\big\downarrow & & \big\downarrow \\ \mathcal{H} & \hookrightarrow & \mathcal{G} \\ {\scriptstyle \binom{1}{0}}\big\downarrow & & \big\downarrow \\ \mathcal{H}' & \rightarrowtail & \mathcal{G}' \end{array} \qquad \begin{array}{ccc} \operatorname{Hom}(\mathcal{F}, \mathcal{H}'') & = & \operatorname{Hom}(\mathcal{F}, \mathcal{H}'') \\ \big\downarrow & & \big\downarrow \\ \operatorname{Hom}(\mathcal{F}, \mathcal{H}) & \xrightarrow{\cong} & \operatorname{Hom}(\mathcal{F}, \mathcal{G}) \\ \big\downarrow & & \big\downarrow \\ \operatorname{Hom}(\mathcal{F}, \mathcal{H}') & \xrightarrow{\cong} & \operatorname{Hom}(\mathcal{F}, \mathcal{G}') \end{array}$$

Therefore the map $\mathcal{H}' \rightarrowtail \mathcal{G}'$ is an $\operatorname{add} \mathcal{F}$-approximation of $\mathcal{G}'$. Since it is mono we have $\mathcal{G}' \in \operatorname{coh}_{\leq\mu(\mathcal{F})+\delta(\mathcal{F})}$ by Corollary 3.5. Hence the short exact sequence $\mathcal{H}'' \hookrightarrow \mathcal{G} \twoheadrightarrow \mathcal{G}'$ splits. Since $\mathcal{G}$ is indecomposable, we either have $\mathcal{G} = \mathcal{H}''$, so f is epi and $\mu(\mathcal{G}) > \mu(\mathcal{F}) + \delta(\mathcal{F})$, or $\mathcal{G} = \mathcal{G}'$, so f is mono and $\mu(\mathcal{G}) \leq \mu(\mathcal{F}) + \delta(\mathcal{F})$. □

3.7. **Corollary.** *Let* $\mathbf{t}$ *be a tube of slope* $\mu \neq \infty$*. Then*

$$\mathcal{G}en\,\mathbf{t} = \operatorname{add}(\mathbf{t} \cup \operatorname{ind}_{>\mu} \mathbb{X}) \text{ and } \mathcal{C}og\,\mathbf{t} = \operatorname{add}(\mathbf{t} \cup \operatorname{ind}_{<\mu} \mathbb{X})$$

3.8. **Proposition.** *Let $\mathcal{F}$ be as above, $\mathcal{G} \in \operatorname{coh}_{>\mu(\mathcal{F})+\delta(\mathcal{F})}$ and $f\colon \mathcal{F}' \twoheadrightarrow \mathcal{G}$ a minimal right $\mathcal{F}$-approximation. Then $\operatorname{Ker} f \xhookrightarrow{\ker f} \mathcal{F}'$ is a minimal left $\mathcal{F}$-approximation of $\operatorname{Ker} f$.*

Proof. We first show that $\ker f$ is an $\mathcal{F}$-approximation, that is, the induced map $\operatorname{Hom}(\mathcal{F}', \mathcal{F}) \longrightarrow \operatorname{Hom}(\operatorname{Ker} f, \mathcal{F})$ is onto. We apply $\operatorname{Hom}(-, \mathcal{F})$ to the short exact sequence $\operatorname{Ker} f \hookrightarrow \mathcal{F}' \twoheadrightarrow \mathcal{G}$, and obtain

$$
\begin{array}{ccccccc}
\operatorname{Hom}(\mathcal{F}', \mathcal{F}) & \longrightarrow & \operatorname{Hom}(\operatorname{Ker} f, \mathcal{F}) & \longrightarrow & \operatorname{Ext}(\mathcal{G}, \mathcal{F}) & \longrightarrow & \operatorname{Ext}(\mathcal{F}', \mathcal{F}) \\
 & & & & \| & & \| \\
 & & & & \operatorname{Hom}(\tau^{-}\mathcal{F}, \mathcal{G})^{*} & \to & \operatorname{Hom}(\tau^{-}\mathcal{F}, \mathcal{F}')^{*}
\end{array}
$$

The lower map is mono, since $\tau^{-}\mathcal{F} \in \operatorname{add} \mathcal{F}$ and f is an $\mathcal{F}$-approximation. Therefore the map $\operatorname{Hom}(\operatorname{Ker} f, \mathcal{F}) \longrightarrow \operatorname{Ext}(\mathcal{G}, \mathcal{F})$ vanishes, and the left map in the diagram is onto as required.

For the minimality, we assume $\mathcal{G}$ to be indecomposable. Assume that the map $\ker f$ factors through a morphism $\mathcal{F}'' \xrightarrow{\varphi} \mathcal{F}'$ in $\operatorname{add} \mathcal{F}$. Then we obtain a cokernel morphism as indicated in the following diagram.

$$
\begin{array}{ccccc}
\operatorname{Ker} f & \xhookrightarrow{\ \ker f\ } & \mathcal{F}' & \xrightarrow{\ f\ } & \mathcal{G} \\
\downarrow & & \| & & \downarrow \\
\mathcal{F}'' & \xrightarrow{\ \varphi\ } & \mathcal{F}' & \twoheadrightarrow & \operatorname{Cok} \varphi
\end{array}
$$

But $\operatorname{Cok} \varphi \in \operatorname{coh}_{\mu(\mathcal{F})} \mathbb{X}$, so any map $\mathcal{G} \longrightarrow \operatorname{Cok} \varphi$ vanishes. Hence φ is onto. However there are no non-spit epimorphisms $\operatorname{add} \mathcal{F} \longrightarrow \mathcal{F}'$ (see construction of the approximation in the proof of Proposition 3.3). Therefore φ is split epi, and hence $\ker f$ is minimal. □

3.9. **Corollary.** *Taking kernels of $\mathcal{F}$-approximations induces an equivalence*

$$\operatorname{coh}_{>\mu(\mathcal{F})+\delta(\mathcal{F})} \mathbb{X} \longrightarrow \operatorname{coh}_{\leq \mu(\mathcal{F})-\delta(\mathcal{F})} \mathbb{X}.$$

We obtain a new proof of the fact that all tubular families are separating. This is “classically”shown using tubular mutations [14].

3.10. **Corollary.** *All tubular families are separating; that is, for any μ, any tube $\mathbf{t}$ of slope μ, any $\mathcal{G} \in \operatorname{coh}_{>\mu} \mathbb{X}$, any $\mathcal{H} \in \operatorname{coh}_{<\mu} \mathbb{X}$, and any $\varphi\colon \mathcal{H} \longrightarrow \mathcal{G}$ there is $\mathcal{F} \in \operatorname{add} \mathbf{t}$ such that φ factors through $\mathcal{F}$.*

Proof. We may assume $\mathcal{G}$ and $\mathcal{H}$ to be indecomposable. Choose $\mathcal{F} \in \operatorname{add} \mathbf{t}$ such that $\tau\mathcal{F} = \mathcal{F}$ and $\delta(\mathcal{F}) < \frac{1}{2} \min\{\mu(\mathcal{G}) - \mu, \mu - \mu(\mathcal{H})\}$. Let $\mathcal{H} \rightarrowtail \mathcal{F}'$

be a minimal left $\mathcal{F}$-approximation of $\mathcal{H}$, and let $\mathcal{C}$ be its cokernel. Then

$$\mu(\mathcal{C}) = \frac{\overline{\chi}(\mathcal{C})}{\mathrm{rk}(\mathcal{C})} = \frac{\overline{\chi}(\mathcal{F}') - \overline{\chi}(\mathcal{H})}{\mathrm{rk}(\mathcal{F}') - \mathrm{rk}(\mathcal{H})} = \frac{\frac{\mathrm{rk}(\mathcal{H})\mu^2 - \overline{\chi}(\mathcal{H})\mu}{\delta(\mathcal{F})} - \overline{\chi}(\mathcal{H})}{\frac{\mathrm{rk}(\mathcal{H})\mu - \overline{\chi}(\mathcal{H})}{\delta(\mathcal{F})} - \mathrm{rk}(\mathcal{H})}$$
$$= \frac{\mu^2 - \mu(\mathcal{H})\mu - \mu(\mathcal{H})\delta(\mathcal{F})}{\mu - \mu(\mathcal{H}) - \delta(\mathcal{F})} = \mu + \delta(\mathcal{F})\frac{\mu - \mu(\mathcal{H})}{\mu - \mu(\mathcal{H}) - \delta(\mathcal{F})}$$
$$< \mu + 2\delta(\mathcal{F}) < \mu(\mathcal{G})$$

Since by Corollary 3.9 the sheaf $\mathcal{C}$ is indecomposable, we have $\mathrm{Ext}(\mathcal{C}, \mathcal{G}) = 0$. Therefore the map $\mathrm{Hom}(\mathcal{F}', \mathcal{G}) \longrightarrow \mathrm{Hom}(\mathcal{K}, \mathcal{G})$ is onto. This means that any map $\mathcal{K} \longrightarrow \mathcal{G}$ factors through $\mathcal{F}'$, as claimed. □

4. Generating the derived category

Since any object in the derived category is a direct sum of stalk complexes, we may restrict our attention to sheaves.

4.1. **Observation.** *Let $\mathcal{F} \in \mathrm{coh}\,\mathbb{X}$ such that all direct summands of $\mathcal{F}$ have the same slope. Then*

$$\langle \mathcal{F} \rangle_\infty \subseteq \langle \cup_{\mathbf{t} \in I} \mathbf{t} \rangle_1 ,$$

where I denotes the set of all tubes which contain a direct summand of $\mathcal{F}$.

If moreover $\tau\mathcal{F} \subseteq \mathrm{add}\,\mathcal{F}$ (so in particular whenever $\mathbb{X}$ is an elliptic curve) then we have equality.

However $\langle \mathcal{F} \rangle_r \subsetneq \langle \cup_{\mathbf{t} \in I} \mathbf{t} \rangle_1$ for any $r \in \mathbb{N}$, so in particular, for any tube $\mathbf{t}$, the triangulated category $\langle \mathbf{t} \rangle_1 = \langle \mathbf{t} \rangle_\infty = D^b(\mathbf{t})$ has no strong generator. In other words: $\dim D^b(\mathbf{t}) = \infty$.

4.2. **Remark.** This shows that the equality $\dim D^b(\mathrm{coh}\,\mathbb{X}) = 1$ does not follow directly from the hereditariness of the abelian category $\mathrm{coh}\,\mathbb{X}$.

The following proposition is essential in showing that certain sheaves from any three given tubes of different slopes generate the derived category in one step.

4.3. **Proposition.** *Let $\mu_1 < \mu_2 < \infty$ and $\mathcal{F}_i \in \mathrm{coh}_{\mu_i}\,\mathbb{X}$ such that $\tau\mathcal{F}_i \in \mathrm{add}\,\mathcal{F}_i$. Assume $\Delta = \mu(\mathcal{F}_2) - \delta(\mathcal{F}_2) - (\mu(\mathcal{F}_1) + \delta(\mathcal{F}_1)) > 0$. Then*

(1) $\mathrm{coh}_{\leq \mu(\mathcal{F}_1) - \delta(\mathcal{F}_1) - \frac{\delta(\mathcal{F}_1)}{\Delta}}\,\mathbb{X} \subseteq \langle \mathcal{F}_1 \oplus \mathcal{F}_2 \rangle_2$,
(2) $\mathrm{coh}_{> \mu(\mathcal{F}_2) + \delta(\mathcal{F}_2) + \frac{\delta(\mathcal{F}_2)}{\Delta}}\,\mathbb{X} \subseteq \langle \mathcal{F}_1 \oplus \mathcal{F}_2 \rangle_2$, *and*
(3) $\mathrm{coh}_{> \mu_1 + \delta(\mathcal{F}_1)}\,\mathbb{X} \cap \mathrm{coh}_{\leq \mu_2 - \delta(\mathcal{F}_2)}\,\mathbb{X} \subseteq \langle \mathcal{F}_1 \oplus \mathcal{F}_2 \rangle_2$.

Proof. For the proof of (1), let $\mathcal{G} \in \mathrm{ind}_{\leq \mu(\mathcal{F}_1) - \delta(\mathcal{F}_1) - \frac{\delta(\mathcal{F}_1)}{\Delta}}\,\mathbb{X}$. By Theorem 3.6, the left $\mathcal{F}_1$-approximation $\mathcal{G} \longrightarrow \mathcal{F}_1'$ of $\mathcal{G}$ is mono, and by Corollary 3.9 the cokernel $\mathcal{C}$ of this approximation is indecomposable. A straightforward calculation, using the formulas in Proposition 3.3, shows

$$\begin{aligned}\mu(\mathcal{C}) = \frac{\overline{\chi}(\mathcal{C})}{\mathrm{rk}(\mathcal{C})} &= \frac{\overline{\chi}(\mathcal{F}_1') - \overline{\chi}(\mathcal{G})}{\mathrm{rk}(\mathcal{F}_1') - \mathrm{rk}(\mathcal{G})} \\ &= \mu(\mathcal{F}_1) + \delta(\mathcal{F}_1) + \frac{\delta(\mathcal{F}_1)}{\mu(\mathcal{F}_1) - \delta(\mathcal{F}_1) - \mu(\mathcal{G})}.\end{aligned}$$

Thus we have

$$\begin{aligned}\mu(\mathcal{C}) &\leq \mu(\mathcal{F}_1) + \delta(\mathcal{F}_1) + \frac{\delta(\mathcal{F}_1)}{\mu(\mathcal{F}_1) - \delta(\mathcal{F}_1) - (\mu(\mathcal{F}_1) - \delta(\mathcal{F}_1) - \frac{\delta(\mathcal{F}_1)}{\Delta})} \\ &= \mu(\mathcal{F}_1) + \delta(\mathcal{F}_1) + \Delta \\ &= \mu(\mathcal{F}_2) - \delta(\mathcal{F}_2).\end{aligned}$$

By Theorem 3.6, this means that $\mathcal{C} \in \mathcal{Cog}\, \mathcal{F}_2$. Hence we have an exact sequence

$$\mathcal{G} \rightarrowtail \mathcal{F}_1' \longrightarrow \mathcal{F}_2' \twoheadrightarrow \mathcal{H},$$

with $\mathcal{F}_i' \in \operatorname{add} \mathcal{F}_i$ and some coherent sheaf $\mathcal{H}$. Since $\operatorname{coh} \mathbb{X}$ is hereditary, any complex in $D^b(\operatorname{coh} \mathbb{X})$ is isomorphic to its homology. Hence the cone of the map $\mathcal{F}_1' \longrightarrow \mathcal{F}_2'$ above is isomorphic to $\mathcal{G}[1] \oplus \mathcal{H}$, and we have a triangle

$$\mathcal{F}_1' \longrightarrow \mathcal{F}_2' \longrightarrow \mathcal{G}[1] \oplus \mathcal{H} \longrightarrow \mathcal{F}_1'[1]$$

in $D^b(\operatorname{coh} \mathbb{X})$. Therefore $\mathcal{G} \in \langle \mathcal{F}_1 \oplus \mathcal{F}_2 \rangle_2$.

The second claim is proven similarly.

For the third claim note that, for any $\mathcal{G} \in \operatorname{coh}_{>\mu_1+\delta(\mathcal{F}_1)} \mathbb{X} \cap \operatorname{coh}_{\leq \mu_2 - \delta(\mathcal{F}_2)} \mathbb{X}$, by Theorem 3.6 there is an epimorphism $\mathcal{F}_1' \twoheadrightarrow \mathcal{G}$ and a monomorphism $\mathcal{G} \rightarrowtail \mathcal{F}_2'$ with $\mathcal{F}_i' \in \operatorname{add} \mathcal{F}_i$. Denoting the kernel of the first map by $\mathcal{K}$ and the cokernel of the latter by $\mathcal{C}$ we obtain the first row and last column of the following diagram. Since $\operatorname{coh} \mathbb{X}$ is hereditary, the induced map $\operatorname{Ext}(\mathcal{C}, \mathcal{F}_1') \twoheadrightarrow \operatorname{Ext}(\mathcal{C}, \mathcal{G})$ is onto, and hence the following diagram may be completed.

$$\begin{array}{ccccc} \mathcal{K} & \hookrightarrow & \mathcal{F}_1' & \twoheadrightarrow & \mathcal{G} \\ \| & & \downarrow & & \downarrow \\ \mathcal{K} & \rightarrowtail & \mathcal{H} & \twoheadrightarrow & \mathcal{F}_2' \\ & & \downarrow & & \downarrow \\ & & \mathcal{C} & = & \mathcal{C} \end{array}$$

The right upper square is exact, so it gives rise to a short exact sequence $\mathcal{F}_1' \rightarrowtail \mathcal{G} \oplus \mathcal{H} \twoheadrightarrow \mathcal{F}_2'$. Hence $\mathcal{G} \in \langle \mathcal{F}_1 \oplus \mathcal{F}_2 \rangle_2$. □

4.4. **Corollary.** *Let* $\mathbf{t}_1$, $\mathbf{t}_2$, *and* $\mathbf{t}_3$ *be tubes of pairwise different slopes. Then there are* $\mathcal{F}_1 \in \operatorname{add} \mathbf{t}_1$, $\mathcal{F}_2 \in \operatorname{add} \mathbf{t}_2$, *and* $\mathcal{F}_3 \in \operatorname{add} \mathbf{t}_3$ *such that*

$$D^b(\operatorname{coh} \mathbb{X}) = \langle \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3 \rangle_2 .$$

Proof. We may assume that none of the slopes is ∞, because otherwise there is by [14] (also see [8, 4.18]) a tubular mutation (an autoequvalence of $D^b(\mathbb{X})$), such that after applying this mutation all slopes are finite. We set $\mu_i = \mu(\mathbf{t}_i)$ $(i = 1, 2, 3)$, and assume $\mu_1 < \mu_2 < \mu_3$.

Let $\mathcal{F}_i \in \mathbf{t}_i$, such that $\operatorname{add} \mathcal{F}_i$ is closed under τ. We denote by $\operatorname{q.LL} \mathcal{F}_i$ the Loewy quasi-length of $\mathcal{F}_i$, that is the quasi-length of the quasi-longest indecomposable direct summand. Moreover we denote by $\mathcal{S}_i$ any quasi-simple in $\mathbf{t}_i$. Then

$$\delta(\mathcal{F}_i) = \frac{1}{\frac{(\operatorname{q.LL} \mathcal{F}_i)^2 p(\mathbf{t}_i)}{\operatorname{q.length} \mathcal{F}_i}} = \underbrace{\frac{1}{p(\mathbf{t}_i) \operatorname{rk}(\mathcal{S}_i)^2}}_{\text{only depends on } \mathbf{t}_i} \frac{1}{\operatorname{q.LL} \mathcal{F}_i}.$$

In particular, $\delta(\mathcal{F}_i)$ gets arbitrarily small when we increase the Loewy quasi-length of $\mathcal{F}_i$.

By Proposition 4.3 we have

(1) $\operatorname{coh}_{\leq \mu_2 - \delta(\mathcal{F}_2) - \frac{\delta(\mathcal{F}_2)}{\mu_2 - \delta(\mathcal{F}_2) - \mu_3 - \delta(\mathcal{F}_3)}} \mathbb{X} \subseteq \langle \mathcal{F}_2 \oplus \mathcal{F}_3 \rangle_2$. Hence, if $\delta(\mathcal{F}_2)$ and $\delta(\mathcal{F}_3)$ are sufficiently small, or equivalently, if $\mathcal{F}_2$ and $\mathcal{F}_3$ have sufficiently large Loewy quasi-length, then

$$\operatorname{coh}_{\leq \frac{\mu_1 + \mu_2}{2}} \mathbb{X} \subseteq \langle \mathcal{F}_2 \oplus \mathcal{F}_3 \rangle_2 .$$

(2) Similarly, if $\mathcal{F}_1$ and $\mathcal{F}_2$ have sufficient Loewy quasi-length, then

$$\operatorname{coh}_{> \frac{\mu_2 + \mu_3}{2}} \mathbb{X} \subseteq \langle \mathcal{F}_1 \oplus \mathcal{F}_2 \rangle_2 ,$$

(3) and if $\mathcal{F}_1$ and $\mathcal{F}_3$ have large enough Loewy quasi-length

$$\operatorname{coh}_{> \frac{\mu_1 + \mu_2}{2}} \mathbb{X} \cap \operatorname{coh}_{\leq \frac{\mu_2 + \mu_3}{2}} \mathbb{X} \subseteq \langle \mathcal{F}_1 \oplus \mathcal{F}_3 \rangle_2 .$$

This covers all of $\operatorname{coh} \mathbb{X}$. Since any object in the derived category is the shift of a sheaf, we have also shown $\langle \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_2 \rangle_2 = D^b(\operatorname{coh} \mathbb{X})$ for any $\mathcal{F}_i \in \mathbf{t}_i$ closed under τ and of sufficient Loewy quasi-length. □

4.5. **Corollary.**

$$\dim D^b(\operatorname{coh} \mathbb{X}) = 1.$$

4.6. **Example.** Let $\mathbb{X}$ be an elliptic curve, and let $\mathcal{L}_1$ and $\mathcal{L}_2$ be quasi-simple sheaves of slope -3 and 3 respectively (then they are line bundles). Then

$$\langle \mathcal{L}_1 \oplus \mathcal{O} \oplus \mathcal{L}_2 \rangle_2 = D^b(\operatorname{coh} \mathbb{X}).$$

Proof. For any line bundle $\mathcal{L}$ over $\mathbb{X}$ we have $\delta(\mathcal{L}) = 1$. □

We end this paper by studying the possibility of generating the derived category from indecomposable sheaves with only two different slopes.

4.7. **Corollary.** *Let $\mu_1 < \mu_2$. Then $\langle \operatorname{ind}_{\mu_1} \mathbb{X} \cup \operatorname{ind}_{\mu_2} \mathbb{X} \rangle_2 = D^b(\operatorname{coh} \mathbb{X})$.*

4.8. **Corollary.** *Let* $\mu_1 < \mu_2$ *and* $\mathcal{F}_i \in \operatorname{coh}_{\mu_i} \mathbb{X}$ *closed under* τ. *Then* $\langle \mathcal{F}_1 \oplus \mathcal{F}_2 \rangle_\infty = D^b(\operatorname{coh} \mathbb{X})$, *and therefore, by Observation 2.8, there is* n *such that* $\langle \mathcal{F}_1 \oplus \mathcal{F}_2 \rangle_n = D^b(\operatorname{coh} \mathbb{X})$.

4.9. **Proposition.** *Let* $\mu_1 < \mu_2$ *and* $\mathcal{F}_i \in \operatorname{coh}_{\mu_i} \mathbb{X}$. *Then* $\langle \mathcal{F}_1 \oplus \mathcal{F}_2 \rangle_2 \subsetneq D^b(\operatorname{coh} \mathbb{X})$.

Proof. Let $\mathcal{G} \in \operatorname{coh}_{\mu_2} \mathbb{X}$, such that $\mathcal{G}$ does not lie in the same tube as any of the direct summands of $\mathcal{F}_2$. Assume $\mathcal{G} \in \langle \mathcal{F}_1 \oplus \mathcal{F}_2 \rangle_2 \subsetneq D^b(\operatorname{coh} \mathbb{X})$. That means there is a triangle $\mathcal{F}' \longrightarrow \mathcal{F}'' \longrightarrow \mathcal{G} \oplus \mathcal{C} \longrightarrow \mathcal{F}'[1]$ for some $\mathcal{F}'$, $\mathcal{F}'' \in \langle \mathcal{F}_1 \oplus \mathcal{F}_2 \rangle$ and $\mathcal{C} \in D^b(\operatorname{coh} \mathbb{X})$. We can write $\mathcal{F}' = \mathcal{F}_1' \oplus \mathcal{F}_2'$ and $\mathcal{F}'' = \mathcal{F}_1'' \oplus \mathcal{F}_2''$ with $\mathcal{F}_1', \mathcal{F}_1'' \in \langle \mathcal{F}_1 \rangle$ and $\mathcal{F}_2', \mathcal{F}_2'' \in \langle \mathcal{F}_2 \rangle$.

By the octahedral axiom we obtain the following commutative diagram.

$$\begin{array}{ccccccc} & & \mathcal{F}_2'' & = & \mathcal{F}_2'' & & \\ & & \downarrow{\scriptstyle \text{incl}} & & \downarrow & & \\ \mathcal{F}' & \longrightarrow & \mathcal{F}'' & \longrightarrow & \mathcal{G} \oplus \mathcal{C} & \longrightarrow & \mathcal{F}'[1] \\ \| & & \downarrow{\scriptstyle \text{proj}} & & \downarrow & & \| \\ \mathcal{F}' & \longrightarrow & \mathcal{F}_1'' & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{F}'[1] \\ & & \downarrow{\scriptstyle 0} & & \downarrow & & \\ & & \mathcal{F}_2''[1] & = & \mathcal{F}_2''[1] & & \end{array}$$

Applying $\operatorname{Hom}(-, \mathcal{G})$ to the right vertical triangle, we see that the map $\operatorname{Hom}(\mathcal{H}, \mathcal{G}) \longrightarrow \operatorname{Hom}(\mathcal{G} \oplus \mathcal{C}, \mathcal{G})$ is onto, and hence that $\mathcal{G}$ is a direct summand of $\mathcal{H}$. Therefore we may assume $\mathcal{F}'' \in \langle \mathcal{F}_1 \rangle$, and similarly one may assume $\mathcal{F}' \in \langle \mathcal{F}_1 \rangle$. This is a contradiction, since clearly $\langle \mathcal{F}_1 \rangle_\infty \subset \operatorname{add}\{(\operatorname{ind}_{\mu_1} \mathbb{X})[i] \mid i \in \mathbb{Z}\}$. Therefore $\mathcal{G} \notin \langle \mathcal{F}_1 \oplus \mathcal{F}_2 \rangle_2$. □

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