Lower bounds for Auslander's representation dimension

Steffen Oppermann

Mathematisches Institut der Universität zu Köln, Weyertal 86-90,
50931 Köln, Germany
current address: Institutt for Matematiske Fag, NTNU, 7491 Trondheim, Norway
(e-mail: steffen.oppermann@math.ntnu.no)

Abstract. The representation dimension is an invariant introduced by Auslander to measure how far a representation infinite algebra is from being representation finite. In 2005, Rouquier has given the first examples of algebras of representation dimension greater than three. Here, we give the first general method for establishing lower bounds for the representation dimension of given algebras or families of algebras. The classes of algebras for which we explicitly apply this method include (but do not restrict to) most of the previous examples of algebras of large representation dimension, for some of which the lower bound is improved to the correct value.

Introduction

The representation dimension of a finite dimensional algebra was introduced by Auslander in his Queen Mary College notes [1]. Auslander has shown that an algebra is of finite representation type if and only if its representation dimension is at most two. In general, he expected that the representation dimension should measure how far an algebra is from being of finite representation type.

In [14] (see also [13, 15] and the appendix here), Iyama showed that the representation dimension of a finite dimensional algebra is always finite.

The first example of an algebra with representation dimension strictly greater than three has been given by Rouquier in his article on the representation dimension of exterior algebras [21]. In this paper he has shown that it is possible to use the dimension of the derived category or, in the case of self-injective algebras, of the stable module category, to obtain lower bounds for the representation dimension. Using this, he has proven that the representation dimension of the exterior algebra of an n-dimensional vector space is $n+1$.

A second class of examples has been given by Krause and Kussin. In [17] they have shown that the representation dimension of the algebras kQ/I, with Q and I as in the case $l = n$ of $(\star)$ on page 3, is at least $n-1$.

In [19] the author has shown that the representation dimension of an elementary abelian group is at least its rank plus one.

Avramov and Iyengar [5], using techniques from [4], have announced that the dimension of the stable derived category of a complete intersection local ring R is at least the codimension of R minus one. As a corollary they deduce that when in addition the ring is artin, its representation dimension is at least the embedding dimension plus one. In particular, the representation dimension of $k[x_1, ..., x_n]/(x_1^{c_1}, ..., x_n^{c_n})$ is at least $n+1$, generalizing the results in [19].

Here we give a more general method to find lower bounds for the representation dimension of classes of algebras. The main ingredients are as follows:

We extend Rouquier's definition of dimension of a triangulated category to subcategories. This will allow us to find better lower bounds than by looking only at the dimension of the derived category. In many examples we will even be able to show that the representation dimension is strictly larger than the dimension of the derived category. In particular we will be able to improve Krause and Kussin's bound to $n + 1$, which will then be shown to be the precise value.

To find a lower bound to the representation dimension of a finite dimensional k-algebra Λ, we have to look at an entire family of modules at once. In this paper we will assume this family to come from one $\Lambda \otimes_k R$-lattice L, where R is a finitely generated commutative domain over k. That is, the modules in the family we consider are the modules of the form $L \otimes_R X$ for R-modules X of finite length, or, in other words, we look at the image of the category of R-modules of finite length under the functor

$$L \otimes_R - : R\text{-mod} \longrightarrow \Lambda\text{-Mod}\,.$$

This functor is exact, and therefore also induces maps

$$(L \otimes_R -)_{\mathrm{Ext}^d} : \mathrm{Ext}^d_R(X, Y) \longrightarrow \mathrm{Ext}^d_\Lambda(L \otimes_R X, L \otimes_R Y)$$

for any $X, Y \in R$-mod. With this notation, the main result presented here is

Theorem (Corollary 4.9)**.** *Let L be a $\Lambda \otimes_k R$-lattice. Assume the set*

$$\{\mathfrak{p} \in \operatorname{MaxSpec} R \mid (L \otimes_R -)_{\mathrm{Ext}^d}(\mathrm{Ext}^d_R(R_\mathfrak{p}\text{-f.l.}, R_\mathfrak{p}\text{-f.l.})) \neq 0\}$$

is Zariski dense (here $\operatorname{MaxSpec} R$ *denotes the set of maximal ideals of R, and* $R_\mathfrak{p}$-f.l. *denotes the category of modules over the local ring $R_\mathfrak{p}$ which have finite length). Then*

$$\operatorname{repdim} \Lambda \geq d + 2.$$

We actually prove a refinement of this theorem, which works with complexes of injectives in the derived category (Theorem 1) and a version which is easier to apply to examples (Theorem 2). It turns out that these theorems provide useful lower bounds for the representation dimension in a variety of situations, in many of which we will see that these lower bounds are equal or very close to the correct number. This will be done by exploiting Iyama's result in the appendix.

We reprove Rouquier's result on the representation dimension of the exterior algebra of an n-dimensional vector space and generalize it to the quotient of the exterior algebra modulo the l-th power of the radical (Example 6.1). For $l \neq n$ we can show that the lower bound we find for the representation dimension is the precise value (Example A.6).

We prove that the representation dimension of $k[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^l$ is at least $\min\{l+1, n+1\}$ (Example 6.2). For $n \geq l$ we are able to show that this is the correct number (Example A.9). This result carries over (see Section 7) to algebras of the form kQ/I, with

$$Q = \begin{tikzcd} \overset{1}{\circ} \arrow[r, shift left=3, "x_1"] \arrow[r, shift right=3, "x_n"'] \arrow[r, phantom, "\vdots"] & \overset{2}{\circ} \arrow[r, shift left=3, "x_1"] \arrow[r, shift right=3, "x_n"'] \arrow[r, phantom, "\vdots"] & \overset{3}{\circ} & \cdots & \overset{l-1}{\circ} \arrow[r, shift left=3, "x_1"] \arrow[r, shift right=3, "x_n"'] \arrow[r, phantom, "\vdots"] & \overset{l}{\circ} \end{tikzcd} \text{ and} \qquad (\star)$$

$$I = (x_{n'}x_{n''} - x_{n''}x_{n'} \mid 1 \leq n', n'' \leq n)$$

(Example 7.3). This generalizes the family considered by Krause and Kussin. In particular we improve the lower bound in their case ($l = n$) from $n-1$ to $n+1$, and show that this is the precise value (Example A.8).

One advantage of the theorem presented here is that it is quite well behaved under changes of the algebra. In most of the previous papers an equivalence of the derived or stable module category to some other triangulated category has been used. In that case one did not automatically get any results for similar algebras. With the method presented here it will usually be possible to move results to other algebras with a similar structure (Section 6 and especially Section 7). Especially we will get lower bounds for the representation dimension of algebras depending on parameters in k, not just for discrete families (Examples 7.4, 7.4.1 and 7.5).

We will recall and fix some general notation for our algebras and module categories in the first section.

In the second section we will give definitions of the representation dimension of a finite dimensional algebra (due to Auslander [1]) and the dimension of a triangulated category (due to Rouquier [21, 22]). We will generalize the latter to subcategories of triangulated categories. Finally we will prove inequalities between these dimensions to be used in the rest of this paper.

The third section will be used to study the vanishing of extensions over polynomial rings or integral quotients of such. We need to do so to be able to transfer these properties to the module categories of other algebras with the help of tensor functors in Section 4.

The fourth section will be used to prove the main theorem (Theorem 1). To do so, we will look at a pair of adjoint functors between the derived categories of the module category we are interested in and the category of finite length modules over the commutative ring we studied in Section 3.

In the fifth section we will further analyze one special case of the main theorem. This will lead to Theorem 2, a reformulation of the main theorem in this case, which looks more technical but is easier to apply to examples.

The sixth section will be used to show how the results apply to concrete algebras. In particular we will generalize Rouquier's result (Example 6.1) and get a lower bound for the representation dimension of quotients of polynomial rings (Theorem 3).

In the seventh section we will show that the assumptions of the main theorem are preserved under certain coverings. Therefore we get results on variations of the examples presented in Section 5. In particular we will improve the result of Krause and Kussin (Example 7.3) and we get results on larger families (families indexed by continuous parameters, not just discrete families) of algebras (Examples 7.4 and 7.5).

The appendix contains results mainly due to Iyama. We apply the general upper bound he established for the representation dimension of finite dimensional algebras to the examples we considered in Sections 6 and 7. It turns out that in most cases either the lower bound is equal to the upper bound or the difference is very small.

Acknowledgements. I wish to thank my supervisor, Steffen Koenig, for support, and for many helpful suggestions, and Osamu Iyama for providing and for permission to include most of the ideas of the appendix and for remarks on the rest of the paper.

1 Notation

Throughout this paper, k will denote a commutative (not necessarily algebraically closed) field. By a finite dimensional algebra we mean an associative algebra Λ over k, which has finite dimension as a k-vector space. For any associative ring Λ we denote by Λ-mod the category of finitely presented left Λ-modules. In case Λ is a finite dimensional algebra, this is just the category of Λ-modules of finite k-dimension. We denote the Jacobson radical of an algebra Λ by J_Λ, or, if there is no chance of confusion, simply by J. The global dimension of an algebra Λ will be denoted by $\operatorname{gld}\Lambda$.

For $M \in \Lambda$-mod we denote by $\operatorname{add} M$ the full subcategory of Λ-mod consisting of all direct summands of finite direct sums of copies of M. Moreover, if Λ is a k-algebra, we denote by $M^* = \operatorname{Hom}_k(M, k) \in \Lambda^{\mathrm{op}}$-mod the k-dual of M.

We denote by Λ-proj and Λ-inj the categories of finitely presented projective and injective Λ-modules, respectively. Note that Λ-proj $= \operatorname{add}\Lambda$, and, if Λ is a finite dimensional algebra, Λ-inj $= \operatorname{add}\Lambda^*$.

For an additive category $\mathcal{A}$ (in this paper typically one of Λ-mod, Λ-proj, Λ-inj) we denote by $K(\mathcal{A})$ the homotopy category of complexes in $\mathcal{A}$. We denote by $K^b(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^{[a_1,a_2]}(\mathcal{A})$ the subcategories of complexes which are bounded, bounded above (that is to the right), and concentrated in degrees $a_1, \ldots, a_2$, respectively. We denote by [1] the shift, that is the functor moving any complex one step to the left, and replacing all differentials by their negatives (see [16, Section 1]).

We denote by $D(\Lambda\text{-mod})$ the derived category of Λ-mod, that is the category obtained from $K(\Lambda\text{-mod})$ by localizing at the subcategory of quasi-isomorphisms. The variations D^b, D^-, $D^{[a_1,a_2]}$ and the shift [1] are defined as they are in the case of the homotopy category.

In all the categories above we compose maps from left to right, that is fg means first applying f and then g.

2 Dimensions

We need to introduce a few notions of dimension and to determine the relations between them. The representation dimension of an artin algebra has been introduced by Auslander [1]. Rouquier has introduced the dimension of a triangulated category [21, 22], and he [21, 22] and Krause and Kussin [17] (see also Christensen [8]) have proven the inequalities involving these dimensions presented here. The principal new results in this section are the extension of Rouquier's definition to subcategories of triangulated categories (Definition 2.7), in particular to module categories, and the inequalities involving this new notion of dimension (Lemmas 2.9 and 2.13).

We start by recalling the definition of representation dimension. Note that while this is not the original definition Auslander has given in [1], it follows from [1] that this definition is equivalent to his, except in case Λ is semisimple.

2.1 Definition. Let Λ be a finite dimensional algebra over a field. The *representation dimension* of Λ is defined to be

$$\operatorname{repdim}\Lambda = \min\{\operatorname{gld}\operatorname{End}_\Lambda(M) \mid M \text{ generates and cogenerates } \Lambda\text{-mod}\}.$$

The condition that the module M generates (cogenerates) Λ-mod means that any module is a quotient (submodule) of a finite direct sum of copies of M, or, equivalently, that any indecomposable projective (injective) Λ-module is isomorphic to a direct summand of M.

Auslander's expectation was that the representation dimension should measure how far an algebra is from having finite representation type. This is motivated by the following result:

Theorem (Auslander [1])**.** *Let Λ be a finite dimensional algebra. Then Λ is of finite representation type if and only if* $\operatorname{repdim}\Lambda \leq 2$.

2.2 Definition. Let Λ be a finite dimensional algebra and $M \in \Lambda$-mod. Then

- the *M-resolution dimension* of a module $X \in \Lambda$-mod is defined to be

$$\begin{aligned} M\text{-resol.dim}\, X = \min\{n \in \mathbb{N} \mid{} & \text{there is a complex} \\ & 0 \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_0 \longrightarrow X \longrightarrow 0 \\ & \text{with } M_i \in \operatorname{add} M \text{ such that the induced complex} \\ & 0 \longrightarrow \operatorname{Hom}_\Lambda(M, M_n) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_\Lambda(M, X) \longrightarrow 0 \\ & \text{is exact}\}, \end{aligned}$$

  (here and in the following definitions we set $\min \emptyset = \infty$.)

- the *M-resolution dimension* of a subcategory $\mathcal{X} \subseteq \Lambda$-mod is defined to be
$$M\text{-resol.dim}\, \mathcal{X} = \sup\{M\text{-resol.dim}\, X \mid X \in \operatorname{Ob}\mathcal{X}\}.$$

2.3 Remark. If M is a generator then the complex

$$0 \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_0 \longrightarrow X \longrightarrow 0$$

in the definition above is an exact sequence.

This motivates the following weak version of 2.2:

2.4 Definition. Let Λ be a finite dimensional algebra and $M \in \Lambda$-mod. Then

- the *weak M-resolution dimension* of a module $X \in \Lambda$-mod is defined to be

$$\begin{aligned}M\text{-wresol.dim}\, X = \min\{n \in \mathbb{N} \mid & \text{ there is an exact sequence} \\ & 0 \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_0 \longrightarrow X \longrightarrow 0 \\ & \text{with } M_i \in \operatorname{add} M\},\end{aligned}$$

- the *weak M-resolution dimension* of a subcategory $\mathcal{X} \subseteq \Lambda$-mod is defined to be

$$M\text{-wresol.dim}\, \mathcal{X} = \sup\{M\text{-wresol.dim}\, X \mid X \in \mathcal{O}b\, \mathcal{X}\},$$

- the *weak resolution dimension* of a subcategory $\mathcal{X} \subseteq \Lambda$-mod is defined to be
$$\text{wresol.dim}\, \mathcal{X} = \min_{M \in \Lambda\text{-mod}} M\text{-wresol.dim}\, \mathcal{X}.$$

2.5 Lemma ([10, Lemma 2.1])**.** *Let Λ be a finite dimensional, non-semisimple algebra. Let $M \in \Lambda$-mod be generator and cogenerator. Then*

$$\operatorname{gld} \operatorname{End}_\Lambda(M) = M\text{-resol.dim}(\Lambda\text{-mod}) + 2.$$

In particular

$$\operatorname{repdim} \Lambda = \min_{\substack{M \text{ generator} \\ \text{and cogenerator}}} M\text{-resol.dim}(\Lambda\text{-mod}) + 2.$$

2.6 Remark. If M is a generator then Remark 2.3 implies that

$$M\text{-wresol.dim}\, X \leq M\text{-resol.dim}\, X$$

and in particular

$$\text{wresol.dim}(\Lambda\text{-mod}) + 2 \leq \operatorname{repdim} \Lambda.$$

Let $\mathcal{T}$ be a triangulated category. We use $*$, $\diamond$ and $\langle - \rangle_n$ as in [21, 22] (see also [6] and [7]). That is, for a subcategory $\mathcal{I}$ of $\mathcal{T}$ we denote by $\langle \mathcal{I} \rangle$ the full subcategory whose objects are direct summands of finite direct sums of shifts of objects in $\mathcal{I}$. For two subcategories $\mathcal{I}_1$ and $\mathcal{I}_2$, we denote by $\mathcal{I}_1 * \mathcal{I}_2$ the full subcategory of extensions between the objects of $\mathcal{I}_2$ and those of $\mathcal{I}_1$. That means that $\mathcal{I}_1 * \mathcal{I}_2$ is the full subcategory of $\mathcal{T}$ with

$$\begin{aligned}\mathcal{O}b\, \mathcal{I}_1 * \mathcal{I}_2 = \{X \in \mathcal{O}b\, \mathcal{T} \mid & \text{ there is a distinguished triangle} \\ & X_1 \longrightarrow X \longrightarrow X_2 \longrightarrow X_1[1] \text{ with } X_i \in \mathcal{I}_i\}.\end{aligned}$$

Now let

$$\mathcal{I}_1 \diamond \mathcal{I}_2 = \langle \mathcal{I}_1 * \mathcal{I}_2 \rangle \, .$$

We set

$$\begin{aligned} \langle \mathcal{I} \rangle_0 &= 0, \\ \langle \mathcal{I} \rangle_1 &= \langle \mathcal{I} \rangle, \text{ and inductively} \\ \langle \mathcal{I} \rangle_{n+1} &= \langle \mathcal{I} \rangle_n \diamond \langle \mathcal{I} \rangle. \end{aligned}$$

2.7 Definition. Let $\mathcal{T}$ be a triangulated category, $\mathcal{C} \subseteq \mathcal{T}$ a subcategory. We define the *dimension* of $\mathcal{C}$ to be

$$\dim_{\mathcal{T}} \mathcal{C} = \min\{n \in \mathbb{N} \mid \exists M \in \mathcal{O}b\, \mathcal{T} : \mathcal{C} \subseteq \langle M \rangle_{n+1}\}.$$

Note that for $\mathcal{C} = \mathcal{T}$ this coincides with Rouquier's definition [21, 22] of dimension of a triangulated category.

We will omit the index $\mathcal{T}$ whenever there is no danger of confusion. In particular for $\mathcal{C} = \Lambda$-mod, interpreted as a subcategory of $D^b(\Lambda\text{-mod})$ by identifying modules with complexes concentrated in degree 0, we will write $\dim \Lambda$-mod instead of $\dim_{D^b(\Lambda\text{-mod})} \Lambda$-mod.

The following lemma is an immediate consequence of the definition:

2.8 Lemma. *1. Let $\mathcal{C} \subseteq \mathcal{D}$ be subcategories of a triangulated category $\mathcal{T}$. Then $\dim \mathcal{C} \leq \dim \mathcal{D}$.*

2. Let $\mathtt{F} : \mathcal{T} \longrightarrow \mathcal{T}'$ be a triangulated functor. Let $\mathcal{C} \subseteq \mathcal{T}$. Then $\dim_{\mathcal{T}'} \mathtt{F}(\mathcal{C}) \leq \dim_{\mathcal{T}} \mathcal{C}$.

2.9 Lemma. *Let Λ be a finite dimensional algebra. Then for any $\mathcal{X} \subseteq \Lambda$-mod*

$$\dim \mathcal{X} \leq \operatorname{wresol.dim} \mathcal{X}.$$

Proof. This is an immediate consequence of the fact that short exact sequences in Λ-mod are turned into triangles in $D^b(\Lambda\text{-mod})$. □

2.10 Lemma (a special case of [22, Lemma 7.35])**.** *Let Λ be a finite dimensional algebra. Let $X = (X^i)_{i \in \mathbb{Z}}$ be a complex of Λ-modules, such that all X^i have Loewy length at most n. Then $X \in \langle \Lambda/J\Lambda \rangle_n \subseteq D^b(\Lambda\text{-mod})$.*

In particular

$$\dim D^b(\Lambda\text{-mod}) \leq \operatorname{LL} \Lambda - 1,$$

where LL *denotes the Loewy length.*

2.11 Lemma (see [4, Theorem 5.5])**.** *Let Λ be a finite dimensional algebra. Let $X \in D^b(\Lambda\text{-mod})$ be a complex such that all homology modules have projective dimension at most n. Then $X \in \langle \Lambda \rangle_{n+1} \subseteq D^b(\Lambda\text{-mod})$.*

In particular

$$\dim D^b(\Lambda\text{-mod}) \leq \operatorname{gld} \Lambda.$$

2.12 Corollary ([21, Proposition 3.7] (separable case) and [17, Corollary 3.6])**.** *Let Λ be a finite dimensional algebra. Then*

$$\operatorname{repdim} \Lambda \geq \dim D^b(\Lambda\text{-mod}).$$

Let us illustrate the most important dimensions and inequalities in the following diagram, where a line means that the upper expression is larger than or equal to the lower one.

$$\begin{tikzcd}
\operatorname{repdim}\Lambda \arrow[d, dash, "+2"'] \arrow[drr, dash] & \operatorname{gld}\Lambda \arrow[dl, dash] \arrow[dr, dash] & \mathrm{LL}\,\Lambda - 1 \arrow[d, dash] \\
\text{wresol.dim}\,\Lambda\text{-mod} \arrow[dr, dash] & & \dim D^b(\Lambda\text{-mod}) \arrow[dl, dash] \\
& \dim \Lambda\text{-mod} &
\end{tikzcd} \tag{†}$$

Here we will get (by two) better lower bounds for the representation dimension by using the left path in the above diagram rather than just the inequality $\dim D^b(\Lambda\text{-mod}) \leq \operatorname{repdim} \Lambda$.

Note that, for Λ self-injective, Rouquier [21] also improved the lower bound he obtained for the representation dimension from $\dim D^b(\Lambda\text{-mod})$ to $\dim \Lambda\text{-}\underline{\text{mod}} + 2$ by looking at the dimension of the stable module category $\Lambda\text{-}\underline{\text{mod}}$ rather then at the derived category. The following lemma shows that his improvement is included in ours in that case.

2.13 Lemma. *Let Λ be a self-injective finite dimensional algebra. Then*

$$\dim \Lambda\text{-mod} \geq \dim \Lambda\text{-}\underline{\text{mod}}\,.$$

Proof. The projection functor $D^b(\Lambda\text{-mod}) \longrightarrow D^b(\Lambda\text{-mod})/\Lambda\text{-perf} = \Lambda\text{-}\underline{\text{mod}}$ (see [20, Theorem 2.1]) maps Λ-mod densely to $\Lambda\text{-}\underline{\text{mod}}$. Therefore, by Lemma 2.8, $\dim \Lambda\text{-}\underline{\text{mod}} \leq \dim_{D^b(\Lambda\text{-mod})} \Lambda\text{-mod}$. $\square$

3 Vanishing of extensions over $k[x_1, \dots, x_d]/I$

We fix a field k and $R = k[x_1, \dots, x_d]/I$ with $I \subseteq k[x_1, \dots, x_d]$ a prime ideal. We denote by R-f.l. the category of R-modules of finite length. One main idea of this paper is to look at a family of objects in $D^b(\Lambda\text{-mod})$ by taking a complex G of $\Lambda \otimes_k R$-lattices and looking at the image of the functor

$$G \otimes_R - : D^b(R\text{-f.l.}) \longrightarrow D^b(\Lambda\text{-mod}).$$

The aim of this section is to recall some properties of R-f.l., which will then in the next section be used to study the image of $G \otimes_R -$ in $D^b(\Lambda\text{-mod})$. We will prove Proposition 3.3, which says that for any $M \in D^-(R\text{-mod})$ there is an open subset of blocks of R-f.l., such that for any block in this open subset the homomorphisms from M to this block annihilate all extensions in the block (the blocks of R-f.l. are indexed by maximal ideals of R, so the set of blocks carries Zariski topology).

We denote by $\operatorname{MaxSpec} R$ the set of maximal ideals of R, with Zariski topology. For $\mathfrak{p} \in \operatorname{MaxSpec} R$ we denote by $R_\mathfrak{p}$-f.l. the category of modules of finite length over the localization of R at $\mathfrak{p}$. This is the full subcategory of R-f.l. whose objects are all iterated extensions of the simple module $R/\mathfrak{p}$. This yields a block decomposition

$$R\text{-f.l.} = \coprod_{\mathfrak{p} \in \operatorname{MaxSpec} R} R_\mathfrak{p}\text{-f.l.}$$

That is R-f.l. decomposes as a coproduct of categories, in other words any module of finite length is the direct sum of modules in the different $R_\mathfrak{p}$-f.l., and for $\mathfrak{p} \neq \mathfrak{q}$ there are no non-zero morphisms from $R_\mathfrak{p}$-f.l. to $R_\mathfrak{q}$-f.l..

We will need the following result from commutative algebra:

3.1 Lemma. *Let R be as above, and M a finitely generated R-module. Then there is a non-empty open subset $\mathcal{U} \subseteq \operatorname{MaxSpec} R$ such that $R_\mathfrak{p} \otimes_R M$ is a free $R_\mathfrak{p}$-module for any $\mathfrak{p} \in \mathcal{U}$.*

Proof. By [18, Theorem 4.10(ii)] the set

$$\widetilde{\mathcal{U}} = \{\mathfrak{p} \in \operatorname{Spec} R \mid R_\mathfrak{p} \otimes_R M \text{ is free over } R_\mathfrak{p}\}$$

is open in $\operatorname{Spec} R$. Since the local ring at the generic point $R_{\{0\}}$ is the quotient field of R, we have $\{0\} \in \widetilde{\mathcal{U}}$, so $\widetilde{\mathcal{U}}$ is non-empty. The set $\mathcal{U} = \widetilde{\mathcal{U}} \cap \operatorname{MaxSpec} R$ is open in $\operatorname{MaxSpec} R$, and to see that it is non-empty it suffices to prove that $\operatorname{MaxSpec} R$ is dense in $\operatorname{Spec} R$. Assume $\operatorname{MaxSpec} R \cap \mathcal{V}_r = \emptyset$ for some basic open set $\mathcal{V}_r = \{\mathfrak{p} \in \operatorname{Spec} R \mid r \notin \mathfrak{p}\}$ with $r \in R$. Then r lies in all maximal ideals, hence in the Jacobson radical J_R. However, since R is finitely generated over k the Jacobson radical coincides with the nilradical (see [18, Theorem 5.5]), and since R is a domain the nilradical is 0. Hence $\mathcal{V}_r = \emptyset$, showing that $\operatorname{MaxSpec} R$ is dense in $\operatorname{Spec} R$. $\square$

3.2 Lemma. *For any $\mathfrak{p} \in \operatorname{Spec} R$ the functor $D(R_\mathfrak{p}\text{-Mod}) \longrightarrow D(R\text{-Mod})$ induced by $R \hookrightarrow R_\mathfrak{p}$ is full and faithful.*

Proof. The forgetful functor $R_\mathfrak{p}\text{-Mod} \longrightarrow R\text{-Mod}$ has the left adjoint $R_\mathfrak{p} \otimes_R -$. Since both functors are exact they also form an adjoint pair on the corresponding derived categories. Hence for any $X \in D(R\text{-Mod})$ and $Y \in D(R_\mathfrak{p}\text{-Mod})$ we have

$$\operatorname{Hom}_{D(R_\mathfrak{p}\text{-Mod})}(R_\mathfrak{p} \otimes_R X, Y) = \operatorname{Hom}_{D(R\text{-Mod})}(X, Y).$$

To complete the proof, note that $R_\mathfrak{p} \otimes_R X = X$ if $X \in D(R_\mathfrak{p}\text{-Mod})$. $\square$

3.3 Proposition. *Let $M \in D^-(R\text{-mod})$. There is a non-empty open set $\mathcal{U} \subset \operatorname{MaxSpec} R$ such that for any $\mathfrak{p} \in \mathcal{U}$ and any $X_1, X_2 \in R_\mathfrak{p}\text{-mod}$*

$$\operatorname{Hom}_{D^-(R\text{-mod})}(M, X_1) \operatorname{Hom}_{D^-(R\text{-mod})}(X_1, X_2[1]) = 0.$$

Proof. Since there are projective resolutions in R-mod we may assume that M is represented by a right bounded complex of projectives. Then $\operatorname{Hom}_{D^-(R\text{-mod})}(M, X) = \operatorname{Hom}_{K^-(R\text{-mod})}(M, X)$ for any $X \in K^-(R\text{-mod})$. Any morphism from M to X_1 factors through $\tau^{\geq 0} M$, where $\tau^{\geq 0} M$ is the truncated complex as illustrated in the following diagram.

$$\begin{tikzcd}
M: \arrow[d] & \cdots \arrow[r] & M^{-1} \arrow[r, "\partial"] \arrow[d] & M^0 \arrow[r] \arrow[d] & M^1 \arrow[r] \arrow[d, equal] & \cdots \\
\tau^{\geq 0} M: & \cdots \arrow[r] & 0 \arrow[r] & M^0/\operatorname{Im}\partial \arrow[r] & M^1 \arrow[r] & \cdots
\end{tikzcd}$$

Further, since X_1 in an $R_\mathfrak{p}$-module, any map $\tau^{\geq 0} M \longrightarrow X_1$ factors through $R_\mathfrak{p} \otimes_R \tau^{\geq 0} M$. By Lemma 3.1 there is a non-empty open subset $\mathcal{U} \subseteq \operatorname{MaxSpec} R$ such that $R_\mathfrak{p} \otimes_R M^0/\operatorname{Im}\partial$ is free (as $R_\mathfrak{p}$-module) for any $\mathfrak{p} \in \mathcal{U}$. Then the complex representing $R_\mathfrak{p} \otimes_R \tau^{\geq 0} M$ has only projective terms, and hence $\operatorname{Hom}_{D^b(R_\mathfrak{p}\text{-mod})}(R_\mathfrak{p} \otimes_R \tau^{\geq 0} M, X_2[1]) = \operatorname{Hom}_{K^b(R_\mathfrak{p}\text{-mod})}(R_\mathfrak{p} \otimes_R \tau^{\geq 0} M, X_2[1]) = 0$. The claim of the proposition now follows from Lemma 3.2. $\square$

4 The main theorem

In this section we will state and prove our main theorem. We keep R fixed as in Section 3, and also fix a finite dimensional k-algebra Λ.

One ingredient is the following lemma:

4.1 Lemma (special case of [22, Lemma 4.11])**.** *Let $M \in D^b(\Lambda\text{-mod})$, and $d \in \mathbb{N}$. Assume there is a sequence of d morphisms*

$$N_0 \xrightarrow{f_1} N_1 \xrightarrow{f_2} \cdots \xrightarrow{f_d} N_d$$

in $D^b(\Lambda\text{-mod})$, such that $\operatorname{Hom}_{D^b(\Lambda\text{-mod})}(M[i], N_{j-1}) \cdot f_j = 0$ for all $i \in \mathbb{Z}$ and $j \in \{1, \dots, d\}$. Assume further $X \in D^b(\Lambda\text{-mod})$ such that $\operatorname{Hom}_{D^b(\Lambda\text{-mod})}(X, N_0) \cdot f_1 \cdots f_d \neq 0$. Then $X \notin \langle M \rangle_d$.

4.2 Definition. We call a morphism of complexes $f : (A^i, \partial_A^i) \longrightarrow (B^i, \partial_B^i)$ *locally null-homotopic* if for every i there are maps r^i and s^i as indicated in the following diagram, such that $f^i = r^i \partial_B^{i-1} + \partial_A^i s^i$.

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r, "\partial_A^{i-1}"] \arrow[d, "f^{i-1}"'] & A^i \arrow[r, "\partial_A^i"] \arrow[d, "f^i"'] \arrow[dl, dashed, "r^i"'] & A^{i+1} \arrow[r] \arrow[d, "f^{i+1}"'] \arrow[dl, dashed, "s^i"'] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r, "\partial_B^{i-1}"] & B^i \arrow[r, "\partial_B^i"] & B^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

4.3 Lemma. *Let $M \in D^b(\Lambda\text{-mod})$. Assume there is a sequence of morphisms*

$$N_0 \xrightarrow{f_1} N_1 \xrightarrow{f_2} \cdots \xrightarrow{f_d} N_d$$

in $K^b(\Lambda\text{-inj})$, such that $\operatorname{Hom}_{D^b(\Lambda\text{-mod})}(M[i], N_{j-1}) \cdot f_j = 0$ for all $i \in \mathbb{Z}$ and $j \in \{1, \dots, d\}$. Assume further $f_1 \cdots f_d$ is not locally null-homotopic. Then $\Lambda\text{-mod} \not\subseteq \langle M \rangle_d$.

Proof. Assume that $f_1 \cdots f_d$ is not locally null-homotopic in position p. Let

$$Z = Z^p(N_0) = \operatorname{Ker}[(N_0)^p \xrightarrow{\partial_{N_0}} (N_0)^{p+1}]$$

be the p-cocycles of the complex N_0. Then we have a natural map $h : Z[-p] \longrightarrow N_0$. We will show that $hf_1 \cdots f_d$ is not 0. Then the claim follows from Lemma 4.1.

Assume to the contrary that $hf_1 \cdots f_d = 0$, that is, it is null-homotopic as a map of complexes. That means there is a map $\widetilde{r}$ as indicated in the following diagram, such that $\widetilde{r}\partial_{N_d}^{p-1} = \iota(f_1 \cdots f_d)^p$.

$$\begin{tikzcd}
Z[-p] \arrow[d, "h"] & & & Z \arrow[d, "\iota"] \arrow[dddl, "\widetilde{r}"'] & & & \\
N_0 \arrow[dd, "f_1 \cdots f_d"'] & \cdots \arrow[r] & (N_0)^{p-1} \arrow[r] \arrow[dd] & (N_0)^p \arrow[rr] \arrow[dd] \arrow[dr, "\pi"'] \arrow[ddl, dashed, "r"] & & (N_0)^{p+1} \arrow[r] \arrow[dd] \arrow[ddll, dashed, "s"] & \cdots \\
 & & & & I \arrow[ur] \arrow[dl, dashed, "\widetilde{s}"'] & & \\
N_d & \cdots \arrow[r] & (N_d)^{p-1} \arrow[r] & (N_d)^p \arrow[rr] & & (N_d)^{p+1} \arrow[r] & \cdots
\end{tikzcd}$$

Since $(N_d)^{p-1}$ is injective $\widetilde{r}$ extends to a map r as indicated in the diagram. We have $0 = (\widetilde{r} - \iota r)\partial_{N_d}^{p-1} = \iota((f_1 \cdots f_d)^p - r\partial_{N_d}^{p-1})$, so $(f_1 \cdots f_d)^p - r\partial_{N_d}^{p-1}$ factors through $\operatorname{cok} \iota = \pi$, say via $\widetilde{s}$. Since $(N_d)^p$ is injective $\widetilde{s}$ extends to a map s as indicated in the diagram. Thus $f_1 \cdots f_d$ is locally null-homotopic in position p, contradicting the assumption. $\square$

Now assume we are in the following situation:

4.4 Setup. We have a functor $\mathtt{F} : D^b(R\text{-f.l.}) \longrightarrow D^b(\Lambda\text{-mod})$ with the following two properties:

1. for any $i \in \mathbb{Z}$ there are $a_1^i, a_2^i \in \mathbb{Z}$ such that the functor $\mathtt{F}$ maps R-f.l.$[i]$ ($= D^{[-i,-i]}(R\text{-f.l.})$ = the full subcategory consisting of complexes concentrated in degree $-i$) to $K^{[a_1^i, a_2^i]}(\Lambda\text{-inj})$ (seen as subcategory of $D^b(\Lambda\text{-mod})$, and

2. the functor $\mathtt{F}$ admits a left adjoint $\widetilde{\mathtt{F}} : D^b(\Lambda\text{-mod}) \longrightarrow D^-(R\text{-mod})$ in the following sense: there is a natural isomorphism

$$\operatorname{Hom}_{D^b(\Lambda\text{-mod})}(M, \mathtt{F}X) \cong \operatorname{Hom}_{D^-(R\text{-mod})}(\widetilde{\mathtt{F}}M, X)$$

for $M \in D^b(\Lambda\text{-mod})$ and $X \in D^b(R\text{-f.l.})$.

Note that we do not assume $\mathtt{F}$ to be exact, to commute with shifts, or even to be additive. However we will later choose $\mathtt{F}$ to be a tensor functor, which has all these properties.

4.5 Proposition. *Let* $\mathtt{F} : D^b(R\text{-f.l.}) \longrightarrow D^b(\Lambda\text{-mod})$ *be as described above,* $d \in \mathbb{N}$.

(a) *Assume*

$$\{\mathfrak{p} \in \operatorname{MaxSpec} R \mid \mathtt{F}_{\mathrm{Hom}}(\operatorname{Hom}_{D^b(R\text{-mod})}(R_{\mathfrak{p}}\text{-f.l.}, R_{\mathfrak{p}}\text{-f.l.}[d])) \neq 0\}$$

is dense. Here $\mathtt{F}_{\mathrm{Hom}}$ *denotes the map associated to* $\mathtt{F}$ *sending morphisms between two* R*-modules to morphisms between their images. Then*

$$\dim D^b(\Lambda\text{-mod}) \geq d.$$

(b) *Assume*

$$\{\mathfrak{p} \in \operatorname{MaxSpec} R \mid \mathtt{F}_{\mathrm{Hom}}(\operatorname{Hom}_{D^b(R\text{-mod})}(R_{\mathfrak{p}}\text{-f.l.}, R_{\mathfrak{p}}\text{-f.l.}[d])) \text{ contains at least one map of complexes which is not locally null-homotopic}\}$$

is dense (note that since $\mathtt{F}$ *maps* $R_{\mathfrak{p}}$-f.l. *and* $R_{\mathfrak{p}}$-f.l.$[d]$ *to* $K^b(\Lambda\text{-inj})$ *this makes sense). Then*

$$\dim \Lambda\text{-mod} \geq d,$$

and especially

$$\operatorname{repdim} \Lambda \geq d + 2.$$

For the proof we will need the following observation:

4.6 Lemma. *Let* $M, N \in R$-f.l., *and* $\mathbb{E} \in \operatorname{Ext}^d_R(M, N)$. *Then* $\mathbb{E}$ *can be represented by a* $d+1$*-term exact sequence of finite length* R*-modules.*

Proof. We first show the following claim: Let $N \overset{\iota}{\rightarrowtail} E$ be a monomorphism of R-modules, where N has finite length and E is finitely generated. Then there is a quotient $\widetilde{E}$ of E which has finite length, such that the induced map $N \longrightarrow \widetilde{E}$ is still injective.

Let $N \hookrightarrow I(N)$ be the injective envelope of N. It lifts to a map $E \longrightarrow I(N)$ as indicated in the following diagram.

$$\begin{tikzcd} N \arrow[r, tail] \arrow[d, hook] & E \arrow[dl] \arrow[d, two heads] \\ I(N) & \widetilde{E} \arrow[l, tail] \end{tikzcd}$$

We denote the image of the latter map by $\widetilde{E}$. By [18, Theorem 18.4(v)] any finitely generated submodule of the injective envelope of a simple R-module has finite length. Hence, since N has finite length, also any finitely generated submodule of $I(N)$ has finite length. It remains to see that the induced map $N \longrightarrow \widetilde{E}$ is into. This follows from the fact that $N \hookrightarrow I(N)$ factors through this induced map.

Now we are ready to prove the claim of the lemma. We only have to show that the terms of the exact sequence may be chosen to have finite length. In case $d = 1$ this is automatic, so assume $d > 1$. Let

$$N \rightarrowtail E_d \longrightarrow E_{d-1} \longrightarrow \cdots \longrightarrow E_1 \twoheadrightarrow M$$

be any representative of $\mathbb{E}$. We may assume the E_i to be finitely generated, because there is a projective resolution of M consisting of finitely generated modules, and any d-extension is a pushout of any d-step projective resolution. Let $\widetilde{E}_d$ be any finite length quotient of E_d such that the composition $N \rightarrowtail E_d \twoheadrightarrow \widetilde{E}_d$ is still mono (this is possible by our observation above). Then $\mathbb{E}$ is also represented by the second line of the following diagram, where $\widetilde{E}_{d-1}$ denotes the pushout of square to its upper left.

$$\begin{tikzcd}
N \arrow[r, tail] \arrow[d, equal] & E_d \arrow[r] \arrow[d, two heads] & E_{d-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & E_1 \arrow[r, two heads] \arrow[d, equal] & M \arrow[d, equal] \\
N \arrow[r, tail] & \widetilde{E}_d \arrow[r] & \widetilde{E}_{d-1} \arrow[r] & \cdots \arrow[r] & E_1 \arrow[r, two heads] & M
\end{tikzcd}$$

So we can choose the first term to have finite length. Now the claim follows by induction. □

Proof of Proposition 4.5. We want to apply Lemmas 4.1 and 4.3 for (a) and (b) respectively. Therefore let $M \in D^b(\Lambda\text{-mod})$. Assume the homology of M is concentrated in degrees $b_1, \ldots, b_2$. We set $a_1 = \min\{a_1^i \mid 0 \leq i \leq d-1\}$, $a_2 = \max\{a_2^i \mid 0 \leq i \leq d-1\}$, with the a_j^i as in Setup 4.4, and

$$\widehat{M} = \bigoplus_{i=b_1-a_2}^{b_2-a_1} M[i].$$

That is we take the direct sum of all shifts of M, excluding those which cannot have any morphisms to objects in $\mathtt{F}(R\text{-f.l.}[i])$ for any $i \in \{0, \ldots, d-1\}$. We apply Proposition 3.3 to

$$\bigoplus_{i=0}^{d-1} \widetilde{\mathtt{F}}(\widehat{M})[-i].$$

This yields a non-empty open set $\mathcal{U} \subset \operatorname{MaxSpec} R$, such that for any $\mathfrak{p} \in \mathcal{U}$, any $W_1, W_2 \in R_{\mathfrak{p}}\text{-f.l.}$ and any $i \in \{0, \ldots, d-1\}$ we have

$$\operatorname{Hom}_{D^-(R\text{-mod})}(\widetilde{\mathtt{F}}(\widehat{M}), W_1[i]) \cdot \operatorname{Hom}_{D^-(R\text{-mod})}(W_1[i], W_2[i+1]) = 0.$$

Choose $\mathfrak{p}$ in the intersection of $\mathcal{U}$ with the subset of MaxSpec R described in the hypothesis of the proposition we are proving (that is, we intersect with the set in the hypothesis for part (a) in order to prove (a), and with the set in the hypothesis for part (b) in order to prove (b)). In both cases this is possible by assumption, since $\mathcal{U}$ is non-empty and open.

Now choose an element f of $\operatorname{Hom}_{D^b(R\text{-mod})}(R_{\mathfrak{p}}\text{-f.l.}, R_{\mathfrak{p}}\text{-f.l.}[d])$ which is not mapped to 0 by $\mathtt{F}$. For the proof of (b) choose f such that $\mathtt{F}f$ is not locally null-homotopic. By Lemma 4.6 the morphism f can be decomposed into a product

$$f = f_1 \cdot f_2 \cdots f_d$$

with $f_i \in \operatorname{Hom}_{D^b(R\text{-mod})}(R_{\mathfrak{p}}\text{-f.l.}[i-1], R_{\mathfrak{p}}\text{-f.l.}[i])$, say $f_i : W_{i-1}[i-1] \longrightarrow W_i[i]$. By assumption on $\mathfrak{p}$ we have

$$\operatorname{Hom}_{D^-(R\text{-mod})}(\widetilde{\mathtt{F}}(\widehat{M}), W_{i-1}[i-1]) \cdot f_i = 0.$$

Now we apply $\mathtt{F}$ to f_i and the adjunction isomorphism to the Hom-set. That yields

$$\operatorname{Hom}_{D^b(\Lambda\text{-mod})}(\widehat{M}, \mathtt{F}(W_{i-1}[i-1])) \cdot \mathtt{F}(f_i) = 0.$$

By construction of $\widehat{M}$ this means

$$\operatorname{Hom}_{D^b(\Lambda\text{-mod})}(M[j], \mathtt{F}(W_{i-1}[i-1])) \cdot \mathtt{F}(f_i) = 0 \qquad \forall j \in \mathbb{Z}.$$

Now apply Lemma 4.1 for the proof of (a) and Lemma 4.3 for the proof of (b). $\square$

4.7 Definition. We define $\Lambda \otimes_k R$-lat to be the full subcategory of $\Lambda \otimes_k R$-mod in which the objects are projective as R-modules. We denote by $\operatorname{Inj}(\Lambda \otimes_k R\text{-lat})$ the full subcategory of objects, which are injective with respect to short exact sequences (that is, any short exact sequence which begins in such an object splits).

Note that $\operatorname{Inj}(\Lambda \otimes_k R\text{-lat})$ contains all modules of the form $I \otimes_k R$, with $I \in \Lambda$-inj.

An object $G \in C^b(\operatorname{Inj}(\Lambda \otimes_k R\text{-lat}))$ gives rise to a functor

$$G \otimes_R - : D^b(R\text{-f.l.}) \longrightarrow D^b(\Lambda\text{-mod}).$$

(Since G consists of projective R-modules it is not necessary to derive this functor in order to get a well defined functor between the derived categories.)

Theorem 1. *Let $G \in C^b(\operatorname{Inj}(\Lambda \otimes_k R\text{-lat}))$ and $d \in \mathbb{N}$.*

(a) *Assume*

$$\{\mathfrak{p} \in \operatorname{MaxSpec} R \mid (G \otimes_R -)_{\mathrm{Hom}}(\mathrm{Hom}_{D^b(R\text{-mod})}(R_\mathfrak{p}\text{-f.l.}, R_\mathfrak{p}\text{-f.l.}[d])) \neq 0\}$$

is dense. Then

$$\dim D^b(\Lambda\text{-mod}) \geq d.$$

(b) *Assume*

$$\{\mathfrak{p} \in \operatorname{MaxSpec} R \mid (G \otimes_R -)_{\mathrm{Hom}}(\mathrm{Hom}_{D^b(R\text{-mod})}(R_\mathfrak{p}\text{-f.l.}, R_\mathfrak{p}\text{-f.l.}[d]) \textit{ contains at least one map of complexes which is not locally null-homotopic}\}$$

is dense. Then

$$\dim \Lambda\text{-mod} \geq d,$$

and in particular

$$\operatorname{repdim} \Lambda \geq d + 2.$$

Proof. Clearly we want to apply Proposition 4.5 with $\mathtt{F} = G \otimes_R -$. If G is non-zero only in degrees $c_1, \ldots, c_2$ then $G \otimes_R R\text{-f.l.}[i]$ vanishes outside degrees $c_1 - i, \ldots, c_2 - i$, and hence the first assumption of 4.4 holds. It only remains to show that $\mathtt{F}$ has a left adjoint.

Since G is finitely generated and projective over R it is isomorphic to $\mathrm{Hom}_R(\mathrm{Hom}_R(G, R), R)$ (note that applying $\mathrm{Hom}_R(-, R)$ just means applying $\mathrm{Hom}_R(-, R)$ to every degree). Therefore we have

$$\begin{aligned}
&\mathrm{Hom}_{D^b(\Lambda\text{-mod})}(M, G \otimes_R X) \\
&\qquad \cong \mathrm{Hom}_{D^b(\Lambda\text{-mod})}(M, \mathrm{Hom}_R(\mathrm{Hom}_R(G, R), R) \otimes_R X) \\
&\qquad \cong \mathrm{Hom}_{D^b(\Lambda\text{-mod})}(M, \mathrm{Hom}_R(\mathrm{Hom}_R(G, R), X)) \\
&\qquad \cong \mathrm{Hom}_{D^-(R\text{-mod})}(\mathrm{Hom}_R(G, R) \otimes^L_\Lambda M, X)
\end{aligned}$$

So $\mathrm{Hom}_R(G, R) \otimes^L_\Lambda -$ is the desired adjoint. □

4.8 Remark. Since Λ-inj $\approx$ Λ-proj we may in Theorem 1 alternatively assume $G \in C^b(\Lambda \otimes_k R\text{-proj})$.

Let us now assume that $L \in \Lambda \otimes_k R\text{-lat}$. Then $(L \otimes_R -)$ is an exact functor $R\text{-f.l.} \longrightarrow \Lambda\text{-mod}$. Therefore it also induces maps $(L \otimes_R -)_{\mathrm{Ext}}$ between corresponding Ext-groups.

4.9 Corollary. *Let L be a $\Lambda \otimes R$-lattice, and let $d \in \mathbb{N}$. Assume the set*

$$\{\mathfrak{p} \in \operatorname{MaxSpec} R \mid (L \otimes_R -)_{\mathrm{Ext}^d}(\mathrm{Ext}^d_R(R_\mathfrak{p}\text{-f.l.}, R_\mathfrak{p}\text{-f.l.})) \neq 0\}$$

is dense. Then

$$\dim \Lambda\text{-mod} \geq d,$$

and in particular

$$\operatorname{repdim} \Lambda \geq d+2.$$

Proof. We choose G to consist of the first d terms of an injective resolution of L as $\Lambda \otimes_k R$-lattice (naively truncated, so that it really is a complex of injective lattices). □

5 A practical version of the main theorem

In this section we will treat the following special case: We assume $R = k[x_1, \ldots, x_d]$ and G is a complex of injective lattices of the form $I \otimes_k R$, such that the part of the differential in R is a polynomial of degree one. This setup will be used in the examples.

We denote by $\overline{k}$ the algebraic closure of k. The inclusion $k[x_1, \ldots, x_d] \hookrightarrow \overline{k}[x_1, \ldots, x_d]$ induces a surjection

$$\zeta : \overline{k}^d = \operatorname{MaxSpec} \overline{k}[x_1, \ldots, x_d] \twoheadrightarrow \operatorname{MaxSpec} k[x_1, \ldots, x_d].$$

In particular the ζ-image of a dense subset is dense.

For $(\alpha_1, \ldots, \alpha_d) \in \overline{k}^d = \operatorname{MaxSpec} \overline{k}[x_1, \ldots, x_d]$, we denote by $\widehat{k} = k[\alpha_1, \ldots, \alpha_d]$ the corresponding finite extension of k.

5.1 Corollary. *Let $R = k[x_1, \ldots, x_d]$. Assume $G \in C^b(\operatorname{Inj}(\Lambda \otimes_k R\text{-lat}))$ is of the form*

$$I^0 \otimes_k R \xrightarrow{\partial_0^1 + \sum_{i=1}^d \partial_i^1 x_i} I^1 \otimes_k R \xrightarrow{\partial_0^2 + \sum_{i=1}^d \partial_i^2 x_i} \cdots \xrightarrow{\partial_0^d + \sum_{i=1}^d \partial_i^d x_i} I^d \otimes_k R,$$

with $I^i \in \Lambda$-inj and $\partial_i^j \in \operatorname{Hom}_\Lambda(I^{j-1}, I^j)$. Here ∂_0^j is short for $\partial_0^j \otimes_k 1_R$, and $\partial_i^j x_i$ is short for $\partial_i^j \otimes_k [r \longmapsto r x_i]$. Assume the set

$$\{(\alpha_1, \ldots, \alpha_d) \in \overline{k}^d \mid \textit{the map}$$

$$\begin{tikzcd} & I^0 \otimes_k \widehat{k} \arrow[r, "\partial_0^1 + \sum_{i=1}^d \partial_i^1 \alpha_i"] \arrow[d, "\partial_1^1 \cdots \partial_d^d"] & I^1 \otimes_k \widehat{k} \\ I^{d-1} \otimes_k \widehat{k} \arrow[r, "\partial_0^d + \sum_{i=1}^d \partial_i^d \alpha_i"] & I^d \otimes_k \widehat{k} & \end{tikzcd}$$

is not null-homotopic$\}$

is Zariski dense in $\overline{k}^d$. Then

$$\dim \Lambda\text{-mod} \geq d.$$

Proof. We only need to show that we are in the situation of Theorem 1(b). Assume $(\alpha_1, \ldots, \alpha_d)$ is in the set above. We consider the exact sequences

$$\mathbb{E}_r : \frac{\widehat{k}[x_1, \ldots, x_d]}{\left(\begin{smallmatrix} x_i - \alpha_i \text{ for} \\ i \in \{1, \ldots, d\}\end{smallmatrix}\right)} \xrightarrow{\cdot (x_r - \alpha_r)} \frac{\widehat{k}[x_1, \ldots, x_d]}{\left(\begin{smallmatrix} x_i - \alpha_i \text{ for } i \neq r, \\ (x_r - \alpha_r)^2\end{smallmatrix}\right)} \xrightarrow{\text{proj}} \frac{\widehat{k}[x_1, \ldots, x_d]}{\left(\begin{smallmatrix} x_i - \alpha_i \text{ for} \\ i \in \{1, \ldots, d\}\end{smallmatrix}\right)}$$

of R-modules. Tensoring $\mathbb{E}_r$ with G over R we obtain the following short exact sequence of complexes of Λ-modules.

$$\begin{tikzcd}
I^0 \otimes_k \widehat{k} \arrow[r, "\partial_0^1 + \sum_{i=1}^d \partial_i^1 \alpha_i"] \arrow[d, "(0\ 1)"'] & I^1 \otimes_k \widehat{k} \arrow[r, "\partial_0^2 + \sum_{i=1}^d \partial_i^2 \alpha_i"] \arrow[d, "(0\ 1)"'] & \cdots \arrow[r, "\partial_0^d + \sum_{i=1}^d \partial_i^d \alpha_i"] & I^d \otimes_k \widehat{k} \arrow[d, "(0\ 1)"'] \\
\begin{smallmatrix} I^0 \otimes_k \widehat{k} \oplus \\ I^0 \otimes_k \widehat{k}(x_r - \alpha_r)\end{smallmatrix} \arrow[r, "A^1"] \arrow[d, "\binom{1}{0}"'] & \begin{smallmatrix} I^1 \otimes_k \widehat{k} \oplus \\ I^1 \otimes_k \widehat{k}(x_r - \alpha_r)\end{smallmatrix} \arrow[r, "A^2"] \arrow[d, "\binom{1}{0}"'] & \cdots \arrow[r, "A^d"] & \begin{smallmatrix} I^d \otimes_k \widehat{k} \oplus \\ I^d \otimes_k \widehat{k}(x_r - \alpha_r)\end{smallmatrix} \arrow[d, "\binom{1}{0}"'] \\
I^0 \otimes_k \widehat{k} \arrow[r, "\partial_0^1 + \sum_{i=1}^d \partial_i^1 \alpha_i"] & I^1 \otimes_k \widehat{k} \arrow[r, "\partial_0^2 + \sum_{i=1}^d \partial_i^2 \alpha_i"] & \cdots \arrow[r, "\partial_0^d + \sum_{i=1}^d \partial_i^d \alpha_i"] & I^d \otimes_k \widehat{k}
\end{tikzcd}$$

with $A^j = \begin{pmatrix} \partial_0^j + \sum_{i=1}^d \partial_i^j \alpha_i & \partial_r^j \\ 0 & \partial_0^j + \sum_{i=1}^d \partial_i^j \alpha_i \end{pmatrix}$. The map in the homotopy category corresponding to this extension is

$$\begin{tikzcd}
 & I^0 \otimes_k \widehat{k} \arrow[r, "\partial_0^1 + \sum_{i=1}^d \partial_i^1 \alpha_i"] \arrow[d, "\partial_r^1"'] & I^1 \otimes_k \widehat{k} \arrow[r, "\partial_0^2 + \sum_{i=1}^d \partial_i^2 \alpha_i"] \arrow[d, "\partial_r^2"'] & \cdots \\
I^0 \otimes_k \widehat{k} \arrow[r, "-\partial_0^1 - \sum_{i=1}^d \partial_i^1 \alpha_i"] & I^1 \otimes_k \widehat{k} \arrow[r, "-\partial_0^2 - \sum_{i=1}^d \partial_i^2 \alpha_i"] & I^2 \otimes_k \widehat{k} \arrow[r, "-\partial_0^3 - \sum_{i=1}^d \partial_i^3 \alpha_i"] & \cdots
\end{tikzcd}$$

Now we look at the composition $(G \otimes_R \mathbb{E}_1) \cdots (G \otimes_R \mathbb{E}_d)$. By assumption it is not locally null-homotopic. Therefore $\zeta(\alpha_1, \ldots, \alpha_d)$ is in the set

$$\{\mathfrak{p} \in \operatorname{MaxSpec} R \mid (G \otimes_R -)_{\mathrm{Hom}}(\operatorname{Hom}_{D^b(R\text{-mod})}(R_{\mathfrak{p}}\text{-f.l.}, R_{\mathfrak{p}}\text{-f.l.}[d]) \text{ contains at least one map of complexes which is not locally null-homotopic}\}.$$

Hence this set is dense, so the assumption of Theorem 1(b) is satisfied. $\square$

We now reformulate the hypothesis of Corollary 5.1 in a way which does not require us to use $\Lambda \otimes_k R$-lattices any more. We will only have to find a finite set of morphisms between injective Λ-modules having certain properties.

5.2 Proposition. *For* $0 \leq j \leq d$ *let* $I^j \in \Lambda\text{-inj}$ *and for* $0 \leq i \leq d$ *and* $0 < j \leq d$ *let* $\partial_i^j \in \operatorname{Hom}_\Lambda(I^{j-1}, I^j)$*, such that*

(1) $\forall i, j : \partial_i^j \partial_i^{j+1} = 0$ *and*

(2) $\forall i_1, i_2, j : \partial_{i_1}^j \partial_{i_2}^{j+1} = -\partial_{i_2}^j \partial_{i_1}^{j+1}$.

Assume the set

$$\{(\alpha_1, \ldots, \alpha_d) \in \overline{k}^d \mid \text{for } \widehat{k} = k[\alpha_1, \ldots, \alpha_d] \text{ the map}$$

$$\begin{array}{ccc} & I^0 \otimes_k \widehat{k} & \xrightarrow{\partial_0^1 + \sum_{i=1}^d \partial_i^1 \alpha_i} I^1 \otimes_k \widehat{k} \\ & \downarrow{\scriptstyle \partial_1^1 \cdots \partial_d^d} & \\ I^{d-1} \otimes_k \widehat{k} \xrightarrow{\partial_0^d + \sum_{i=1}^d \partial_i^d \alpha_i} & I^d \otimes_k \widehat{k} & \end{array}$$

$$\text{is not null-homotopic}\}$$

is Zariski dense in $\overline{k}^d$*. Then*

$$\dim \Lambda\text{-mod} \geq d.$$

Proof. We apply Corollary 5.1 to the complex

$$I^0 \otimes_k R \xrightarrow{\partial_0^1 + \sum_{i=1}^d \partial_i^1 x_i} I^1 \otimes_k R \xrightarrow{\partial_0^2 + \sum_{i=1}^d \partial_i^2 x_i} \cdots \xrightarrow{\partial_0^d + \sum_{i=1}^d \partial_i^d x_i} I^d \otimes_k R.$$

Assumptions (1) and (2) of the proposition ensure that this is indeed a complex, that is that the composition of two consecutive morphisms vanishes. □

5.3 Remark. Note that, in Proposition 5.2 above, we have to find out if a morphism of complexes of $\Lambda \otimes_k \widehat{k}$-modules is null-homotopic as a map of complexes of Λ-modules. This seems to be a quite unnatural question. Next we will see that for $\widehat{k}$ separable over k this simplifies to the question whether the map is null-homotopic as a map of complexes of $\Lambda \otimes_k \widehat{k}$-modules.

5.4 Lemma. *Let* $\widehat{k}$ *be a finite separable extension of* k*. A map of complexes of* $\Lambda \otimes_k \widehat{k}$*-modules is (locally) null-homotopic as map of complexes of* Λ*-modules if and only if it is (locally) null-homotopic as a map of complexes of* $\Lambda \otimes_k \widehat{k}$*-modules.*

Proof. The "if"-part is clear.

For the converse let the complexes be (A^i) and (B^i), and the map be (f^i). Assume that there is a Λ-null-homotopy by maps $h^i : A^i \longrightarrow B^{i-1}$.

Since $\widehat{k}$ is separable over k the epimorphism $\widehat{k}\otimes_k\widehat{k} \overset{\pi}{\twoheadrightarrow} \widehat{k}$ of $\widehat{k}$-$\widehat{k}$-bimodules splits ([9, Corollary 69.8]). Let $\iota : \widehat{k} \rightarrowtail \widehat{k} \otimes_k \widehat{k}$ be a morphism of $\widehat{k}$-$\widehat{k}$-bimodules such that $\iota\pi = 1$. This induces maps of $\Lambda \otimes_k \widehat{k}$ modules

$$\begin{tikzcd} A^i \otimes_k \widehat{k} \arrow[r, shift left, "1_{A^i} \otimes_{\widehat{k}} \pi"] & A^i \arrow[l, shift left, "1_{A^i} \otimes_{\widehat{k}} \iota"] \end{tikzcd}$$

and similar for B^i.

Now we replace the h^i by the $\Lambda \otimes_k \widehat{k}$-linear maps

$$\widetilde{h^i} : A^i \xhookrightarrow{1_{A^i} \otimes_{\widehat{k}} \iota} A^i \otimes_k \widehat{k} \xrightarrow{h \otimes_k 1_{\widehat{k}}} B^{i-1} \otimes_k \widehat{k} \xrightarrow{1_{B^{i-1}} \otimes_{\widehat{k}} \pi} B^{i-1}.$$

Note that if $g : X \longrightarrow Y$ is a $\Lambda \otimes_k \widehat{k}$-linear map, then $g(1_Y \otimes_{\widehat{k}} \iota) = (1_X \otimes_{\widehat{k}} \iota)(g \otimes_k 1_{\widehat{k}})$ and $(1_X \otimes_{\widehat{k}} \pi)g = (g \otimes_k 1_{\widehat{k}})(1_Y \otimes_{\widehat{k}} \pi)$. Applying this for the f^i and the differentials of the two complexes it is a straightforward calculation to see that the $\widetilde{h_i}$ also induce a null-homotopy.

The proof for locally null-homotopic is similar. $\square$

We denote by k^{sep} the separable closure of k. Note that $(k^{\mathrm{sep}})^d$ is always dense in $\overline{k}^d$ (If k is infinite then k^d is already dense in $\overline{k}^d$, and otherwise $k^{\mathrm{sep}} = \overline{k}$). Then we obtain the following theorem directly from Proposition 5.2 and Lemma 5.4.

Theorem 2. *For $0 \leq j \leq d$ let $I^j \in \Lambda\text{-inj}$ and for $0 \leq i \leq d$ and $0 < j \leq d$ let $\partial_i^j \in \operatorname{Hom}_\Lambda(I^{j-1}, I^j)$, such that*

(1) $\forall i, j : \partial_i^j \partial_i^{j+1} = 0$ and

(2) $\forall i_1, i_2, j : \partial_{i_1}^j \partial_{i_2}^{j+1} = -\partial_{i_2}^j \partial_{i_1}^{j+1}$.

Assume the set

$$\{(\alpha_1, \dots, \alpha_d) \in (k^{\mathrm{sep}})^d \mid \textit{for } \widehat{k} = k[\alpha_1, \dots, \alpha_d] \textit{ the map}$$

$$\begin{tikzcd} & I^0 \otimes_k \widehat{k} \arrow[r, "\partial_0^1 + \sum_{i=1}^d \partial_i^1 \alpha_i"] \arrow[d, "\partial_1^1 \cdots \partial_d^d"] & I^1 \otimes_k \widehat{k} \\ I^{d-1} \otimes_k \widehat{k} \arrow[r, "\partial_0^d + \sum_{i=1}^d \partial_i^d \alpha_i"] & I^d \otimes_k \widehat{k} & \end{tikzcd}$$

is not null-homotopic as map of complexes over $\Lambda \otimes_k \widehat{k}\}$

is Zariski dense in $(k^{\mathrm{sep}})^d$. Then

$$\dim \Lambda\text{-mod} \geq d.$$

6 Applications

This section is devoted to showing how the results can be applied to some interesting classes of algebras. We will reprove and generalize Rouquier's result on the representation dimension of exterior algebras, and find a general lower bound for the representation dimension of finite dimensional commutative algebras. In the next section we will see that we automatically also get lower bounds for coverings and certain variations of the algebras presented in this section. In the appendix we will find upper bounds for the representation dimension of the algebras we look at in this and the next section. In most cases it will turn out that we have actually identified the representation dimension or that there is only a small number of possible values left.

The examples will consist of families of algebras indexed by l and n, such that l is the Loewy length (we are not very strict about this, see the parenthetical comment on the case $l > n$ in Example 6.1) and n is the number of generators.

As a first example, we consider the exterior algebra, which has been treated by Rouquier [21]. We allow more generally to cut off certain powers of the radical.

6.1 Example. Let $\Lambda_{l,n}$ be the exterior algebra of an n-dimensional vector space modulo the l-th power of the radical ($l > 1$, note that if $l > n$ then the actual value of l does not matter and the Loewy length is $n + 1$). That is

$$\Lambda_{l,n} = k\langle x_1, \dots, x_n\rangle \,/(x_{n'}x_{n''} + x_{n''}x_{n'}, x_{n'}^2, x_{n_1} \cdots x_{n_l} \mid 1 \leq n', n'', n_i \leq n).$$

Then

$$\min\{l-1, n-1\} \leq \dim \Lambda_{l,n}\text{-mod} \leq \dim D^b(\Lambda_{l,n}\text{-mod}) \leq \min\{l-1, n\},$$

and in particular

$$\operatorname{repdim} \Lambda_{l,n} \geq \min\{l+1, n+1\}.$$

Proof. We want to apply Theorem 2. Set $d = \min\{l-1, n-1\}$. Take $I^0 = \cdots = I^d = \Lambda_{l,n}^*$ and ∂_i^j the map induced by right multiplication by x_{i+1}. By definition they fulfill assumptions (1) and (2) of Theorem 2. So consider the diagram

$$\begin{tikzcd}
 & \Lambda_{l,n}^* \otimes_k \widehat{k} \arrow[r, "x_1+\sum_{i=1}^d x_{i+1}\alpha_i"] \arrow[d, "x_2\cdots x_{d+1}"] & \Lambda_{l,n}^* \otimes_k \widehat{k} \\
\Lambda_{l,n}^* \otimes_k \widehat{k} \arrow[r, "x_1+\sum_{i=1}^d x_{i+1}\alpha_i"] & \Lambda_{l,n}^* \otimes_k \widehat{k} &
\end{tikzcd}$$

for any $(\alpha_1, \dots, \alpha_d) \in (k^{\mathrm{sep}})^d$ and $\widehat{k} = k[\alpha_1, \dots, \alpha_d]$. The vertical map of complexes is not null-homotopic. Therefore $\dim \Lambda_{l,n}\text{-mod} \geq d$. The other inequalities are contained in diagram (†) following 2.12. $\square$

Now let us look at truncated polynomial rings.

6.2 Example. Let $\Sigma_{l,n} = k[x_1, \dots, x_n]/(x_1, \dots, x_n)^l$. That is the polynomial ring in n variables modulo all monomials of degree l. Then

$$\min\{l-1, n-1\} \leq \dim \Sigma_{l,n}\text{-mod} \leq \dim D^b(\Sigma_{l,n}\text{-mod}) \leq l-1,$$

and in particular

$$\operatorname{repdim} \Sigma_{l,n} \geq \min\{l+1, n+1\}.$$

Proof. Set $d = \min\{l-1, n-1\}$. Take $I^j = (\Sigma^*_{l,n})^{\binom{d}{j}}$, that is $\binom{d}{j}$ copies of the indecomposable injective module. We assume these copies to be indexed by the subsets of $\{1, \dots, d\}$ having exactly j elements, and write $(\Sigma^*_{l,n})^S$ with $S \subseteq \{1, \dots, d\}$ and $|S| = j$ for the corresponding direct summand of I^j. We define the maps ∂^j_i by giving their components between the direct summands. For ∂^j_0 the component $(\Sigma^*_{l,n})^S \longrightarrow (\Sigma^*_{l,n})^T$ is

$$\begin{cases} 0 & \text{if } S \not\subset T \\ (-1)^{|\{s \in S \mid s < t\}|} x_t & \text{if } S \cup \{t\} = T. \end{cases}$$

For $i > 0$ the component $(\Sigma^*_{l,n})^S \longrightarrow (\Sigma^*_{l,n})^T$ of ∂^j_i is

$$\begin{cases} (-1)^{|\{s \in S \mid s < i\}|} x_n & \text{if } S \cup \{i\} = T \\ 0 & \text{else.} \end{cases}$$

It is a straight forward calculation to verify that these maps fulfill assumptions (1) and (2) of Theorem 2. By induction on d' with $0 \leq d' \leq d$ one can see that the map $\partial^1_1 \cdots \partial^{d'}_{d'}$ is given by its components

$$\left.\begin{array}{ll} 0 & \text{if } S \neq \{1, \dots, d'\} \\ \pm x_n^{d'} & \text{if } S = \{1, \dots, d'\} \end{array}\right\} : (\Sigma^*_{l,n})^{\emptyset} \longrightarrow (\Sigma^*_{l,n})^S.$$

Therefore we consider, for $\alpha \in (k^{\mathrm{sep}})^d$ and $\widehat{k} = k(\alpha)$, the following vertical map of complexes.

$$\begin{array}{ccccc} & & \Sigma^*_{l,n} \otimes_k \widehat{k} & \xrightarrow{(x_1+\alpha_1 x_n, \dots, x_d+\alpha_d x_n)} & (\Sigma^*_{l,n})^d \otimes_k \widehat{k} \\ & & \Big\downarrow {\scriptstyle \pm x_n^d} & & \\ (\Sigma^*_{l,n})^d \otimes_k \widehat{k} & \xrightarrow{\begin{pmatrix} x_1+\alpha_1 x_n \\ -(x_2+\alpha_2 x_n) \\ \vdots \\ \pm(x_d+\alpha_d x_n) \end{pmatrix}} & \Sigma^*_{l,n} \otimes_k \widehat{k} & & \end{array}$$

Clearly it is never null-homotopic. Therefore Theorem 2 can be applied and provides the lower bound for $\dim \Sigma_{l,n}$-mod.

Note that the Loewy length of $\Sigma_{l,n}$ is $\operatorname{LL} \Sigma_{l,n} = l$. Then the other inequalities can be found in diagram (†) following 2.12. □

For an ideal $\mathfrak{a} \subseteq k[x_1, \dots, x_n]$ and $a \in k^n$ we say that $\mathfrak{a}$ has a zero of order l in a if $\mathfrak{a} \subseteq (x_i - a_i \mid 1 \leq i \leq n)^l$. Note that for showing the lower bounds for the three dimensions in Example 6.2 it was only necessary to factor out an ideal which has a zero of order l in 0. Also we can move the zero to any other point by changing the coordinates. Therefore we have shown

Theorem 3. *Let $\mathfrak{a} \subseteq k[x_1, \dots, x_n]$ be an ideal that has a zero of order l. Then*

$$\min\{l-1, n-1\} \leq \dim k[x_1, \dots, x_n]/\mathfrak{a}\text{-mod} \leq \dim D^b(k[x_1, \dots, x_n]/\mathfrak{a}\text{-mod}),$$

and in particular

$$\operatorname{repdim} k[x_1, \dots, x_n]/\mathfrak{a} \geq \min\{l+1, n+1\}.$$

Proof. Say the zero of order l is in $a = (a_1, \dots, a_n)$. Replacing $x_i = y_i + a_i$ we may assume $a = 0$. Then the proof of Example 6.2 carries over word for word (replacing $\Sigma_{l,n}^*$ by $\left(\frac{k[x_1,\dots,x_n]}{\mathfrak{a}}\right)^*$). □

6.3 Remark. Recall the following result due to Avramov and Iyengar [5]:

$$\forall c_1, \dots, c_n > 1 : \operatorname{repdim} k[x_1, \dots, x_n]/(x_1^{c_1}, \dots, x_n^{c_n}) \geq n+1$$

It is worth noting that the result of Theorem 3 intersects their result, where the intersection consists of the cases with $c_1, \dots, c_n \geq n$.

7 Coverings of algebras

The aim of this section is to show that, under certain assumptions, the preconditions of Theorem 1 are invariant under coverings. This result will allow us to transfer the results on the local algebras in the previous section (Examples 6.1 and 6.2) to classes of algebras of finite global dimension. There are many algebras which admit a covering by the same algebra of finite global dimension. This will yield larger families (depending on parameters in k rather than just the discrete parameters l, n) of algebras for which we can find a lower bound for the representation dimension.

We assume Λ to be a finite dimensional graded algebra. Λ being graded means that there is an abelian group A, such that $\Lambda = \oplus_{a \in A} \Lambda_a$ as k-vector space and $\Lambda_{a_1} \cdot \Lambda_{a_2} \subseteq \Lambda_{a_1+a_2}$. Note that the algebras presented in Section 6 as Examples 6.1 and 6.2 are $\mathbb{Z}$ graded by $\deg x_i = 1$.

A graded Λ-module is a Λ-module M with a k-vector space decomposition $M = \oplus_{a\in A} M_a$ such that $\Lambda_{a_1} \cdot M_{a_2} \subseteq M_{a_1+a_2}$. Clearly Λ itself is a graded Λ-module. If M is a graded Λ-modules and $a \in A$, then we denote by $M\langle a\rangle$ the graded Λ-module with $M\langle a\rangle = M$ as Λ-modules, but $M\langle a\rangle_b = M_{a+b}$. For two graded Λ-modules M and N we denote by $\operatorname{Hom}^g_{\mathrm{gr}}(M, N)$ the set of graded homomorphisms of degree g, that is the homomorphisms which map M_a to N_{a+g} for all $a \in A$.

Now let $V \subseteq A$ be a finite subset. We can define a finite dimensional algebra Λ_V by

$$\Lambda_V = (\operatorname{End}^0_{\mathrm{gr}}(\oplus_{v\in V}\Lambda\langle v\rangle))^{\mathrm{op}},$$

Note that $\operatorname{Hom}^0_{\mathrm{gr}}(\Lambda\langle v\rangle\,, \Lambda\langle w\rangle) = \operatorname{Hom}^{w-v}_{\mathrm{gr}}(\Lambda, \Lambda) = \Lambda_{w-v}$. Therefore the algebra Λ_V is the matrix algebra

$$(\Lambda_{w-v})_{\substack{v\in V\\ w\in V}}$$

The indecomposable projective Λ_V-modules are in bijection to the pairs (Q, v) with Q an indecomposable projective Λ-module and $v \in V$, and

$$\operatorname{Hom}_{\Lambda_V}(P_{(Q_1,v_1)}, P_{(Q_2,v_2)}) = \operatorname{Hom}^{v_2-v_1}_{\mathrm{gr}}(Q_1, Q_2).$$

Sending $P_{(Q,v)}$ to Q gives rise to a faithful functor

$$\Lambda_V\text{-proj} \longrightarrow \Lambda\text{-proj}$$

and therefore also to faithful functors

$$\begin{aligned} C^b(\Lambda_V\text{-proj}) &\longrightarrow C^b(\Lambda\text{-proj}), \qquad \text{and} \\ C^b(\Lambda_V \otimes_k R\text{-proj}) &\longrightarrow C^b(\Lambda \otimes_k R\text{-proj}), \end{aligned}$$

which will all be denoted by $\mathtt{C}$.

For $G \in C^b(\Lambda \otimes_k R\text{-proj})$ we set

$$\begin{aligned} d(G) = \max\{d \mid\ & \text{the set} \\ & \{\mathfrak{p} \in \operatorname{MaxSpec} R \mid (G \otimes_R -)_{\mathrm{Hom}}(\operatorname{Hom}_{D^b(R\text{-mod})}(R_{\mathfrak{p}}\text{-f.l.}, R_{\mathfrak{p}}\text{-f.l.}[d]) \\ & \quad \text{contains at least one map of complexes which is not locally null-} \\ & \quad \text{homotopic}\} \\ & \text{is dense}\}. \end{aligned}$$

Then Theorem 1(b) can be restated as follows:

Theorem 1 (b). *Assume $G \in \Lambda \otimes_k R$-proj. Then* $\dim \Lambda\text{-mod} \geq d(G)$.

Our aim is to show that $d(G)$ does not change under certain coverings. Together with the formulation of Theorem 1(b) above this means that we can often establish the same lower bounds for the dimension of the module category of Λ_V that we can show for the dimension of Λ-mod.

7.1 Proposition. *Assume $G \in C^b(\Lambda_V \otimes_k R\text{-proj})$. Then $d(\mathtt{C}G) = d(G)$.*

Proof. Tensoring with $X \in R$-f.l. commutes with $\mathtt{C}$. Let $X_1, X_2 \in R$-f.l. and $\varphi : X_1 \longrightarrow X_2[d]$. Clearly if the map $G \otimes_R X_1 \longrightarrow G \otimes_R X_2[d]$ induced by φ is locally null-homotopic, then so is its image under $\mathtt{C}$.

The idea for the converse is, that all parts of a local null-homotopy which do not respect the grading can be omitted.

More precisely, assume the map

$$\begin{tikzcd}
 & \oplus_i P_{(Q_i,v_i)} \arrow[r, "(\partial_{ij})_{ij}"] \arrow[d, "(f_{ij})_{ij}"'] & \oplus_i P_{(R_i,w_i)} \\
\oplus_i P_{(S_i,x_i)} \arrow[r, "(\partial'_{ij})_{ij}"] & \oplus_i P_{(T_i,y_i)} &
\end{tikzcd}$$

gets null-homotopic by applying $\mathtt{C}$ (here Q_i, R_i, S_i and T_i are indecomposable projective Λ-modules and $v_i, w_i, x_i, y_i \in V$). We want to show that the map then is null-homotopic itself.

By assumption, there are maps $r_{ij} : Q_i \longrightarrow S_j$ and $s_{ij} : R_i \longrightarrow T_j$ as indicated in the following diagram

$$\begin{tikzcd}
 & \oplus_i Q_i \arrow[r, "(\partial_{ij})_{ij}"] \arrow[d, "(f_{ij})_{ij}"'] \arrow[dl, "(r_{ij})_{ij}"'] & \oplus_i R_i \arrow[dl, "(s_{ij})_{ij}"'] \\
\oplus_i S_i \arrow[r, "(\partial'_{ij})_{ij}"] & \oplus_i T_i &
\end{tikzcd}$$

making f null-homotopic.

We can decompose the r_{ij} into $r_{ij} = \sum r_{ij}^g$ with $r_{ij}^g \in \operatorname{Hom}^g_{\mathrm{gr}}(Q_i, S_j)$ and the s_{ij} into $s_{ij} = \sum s_{ij}^g$ with $s_{ij}^g \in \operatorname{Hom}^g_{\mathrm{gr}}(R_i, T_j)$. New recall that the f_{ij}, ∂_{ij} and ∂'_{ij} are graded homomorphisms. Using this fact, it is a straightforward calculation to see that $(r_{ij}^{x_j - v_i})_{ij}$ and $(s_{ij}^{y_j - w_i})_{ij}$ also make f null-homotopic.

The claim now follows from the fact that the $r_{ij}^{x_j - v_i}$ and $s_{ij}^{y_j - w_i}$ are in the image of $\mathtt{C}$. $\square$

We immediately get the following two new examples from Examples 6.1 and 6.2 (the upper bounds for the dimension of the derived category can be read off directly from diagram (†) following 2.12):

7.2 Example (covering of Example 6.1)**.** Let $\Lambda_{l,n}$ be the exterior algebra of an n-dimensional vector space modulo the l-th power of the radical, which was treated in Example 6.1. Let $\widetilde{\Lambda}_{l,n} = (\Lambda_{l,n})_{\{1,\dots,l\}}$, that is the covering with respect to the subset $\{1,\dots,l\} \subset \mathbb{Z}$. Then $\widetilde{\Lambda}_{l,n} = kQ/I$ with

$$Q = \overset{1}{\circ} \;\substack{\xrightarrow{x_1} \\ \vdots \\ \xrightarrow[x_n]{}}\; \overset{2}{\circ} \;\substack{\xrightarrow{x_1} \\ \vdots \\ \xrightarrow[x_n]{}}\; \overset{3}{\circ} \quad \cdots \quad \overset{l-1}{\circ} \;\substack{\xrightarrow{x_1} \\ \vdots \\ \xrightarrow[x_n]{}}\; \overset{l}{\circ} \text{ and}$$
$$I = (x_{n'}x_{n''} + x_{n''}x_{n'}, x_{n'}^2 \mid 1 \leq n', n'' \leq n).$$

Then

$$\min\{l-1, n-1\} \leq \dim \widetilde{\Lambda}_{l,n}\text{-mod} \leq \dim D^b(\widetilde{\Lambda}_{l,n}\text{-mod}) \leq \min\{l-1, n\},$$

and in particular

$$\operatorname{repdim} \widetilde{\Lambda}_{l,n} \geq \min\{l+1, n+1\}.$$

7.3 Example (covering of Example 6.2)**.** Let $\Sigma_{l,n}$ be the truncated polynomial ring as treated in Example 6.2. Let $\widetilde{\Sigma}_{l,n} = (\Sigma_{l,n})_{\{1,\dots,l\}}$. Then $\widetilde{\Sigma}_{l,n} = kQ/I$ with

$$Q = \overset{1}{\circ} \;\substack{\xrightarrow{x_1} \\ \vdots \\ \xrightarrow[x_n]{}}\; \overset{2}{\circ} \;\substack{\xrightarrow{x_1} \\ \vdots \\ \xrightarrow[x_n]{}}\; \overset{3}{\circ} \quad \cdots \quad \overset{l-1}{\circ} \;\substack{\xrightarrow{x_1} \\ \vdots \\ \xrightarrow[x_n]{}}\; \overset{l}{\circ} \text{ and}$$
$$I = (x_{n'}x_{n''} - x_{n''}x_{n'} \mid 1 \leq n', n'' \leq n).$$

Then

$$\min\{l-1, n-1\} \leq \dim \widetilde{\Sigma}_{l,n}\text{-mod} \leq \dim D^b(\widetilde{\Sigma}_{l,n}\text{-mod}) \leq \min\{l-1, n\},$$

and in particular

$$\operatorname{repdim} \widetilde{\Sigma}_{l,n} \geq \min\{l+1, n+1\}.$$

Note that for $l = n$ this is the family of algebras studied by Krause and Kussin [17]. Here we have improved their lower bound for the representation dimension by two.

There can be many graded algebras which have the same covering. We also get the following connection between them:

Let Λ be an A-graded finite dimensional algebra, and let $\alpha : A \longrightarrow \operatorname{Aut}_{\mathrm{gr}} \Lambda$ be a homomorphism of groups. Then we can define a finite dimensional algebra Λ^α by

$$\Lambda^\alpha \cong \Lambda \text{ as } k\text{-vector spaces, and}$$
$$\lambda_1 \cdot_\alpha \lambda_2 = \lambda_1^{\alpha(\deg \lambda_2)} \cdot \lambda_2.$$

It is straight forward to verify that Λ^α is an algebra and that $\Lambda^\alpha_V \cong \Lambda_V$ for any $V \subseteq A$.

In our examples the algebras are $\mathbb{Z}$-graded, and a group homomorphism $\alpha : \mathbb{Z} \longrightarrow \operatorname{Aut}_{\mathrm{gr}} \Lambda$ is determined by $\alpha(1)$. Further note that in Examples 6.1 and 6.2 any automorphism A of the vector space $kx_1 \oplus \cdots \oplus kx_n$ extends uniquely to a graded automorphism of $\Lambda_{l,n}$. Therefore we get the following results:

7.4 Example (from Examples 6.1 and 7.2)**.** Let $A = (a_{ij})$ be an invertible $n \times n$-matrix over k. Let $\Lambda^A_{l,n}$ be the algebra

$$\begin{aligned}\Lambda^A_{l,n} = k\langle x_1, \ldots, x_n\rangle / (&\sum_i a_{n'i} x_{n''} x_i + \sum_i a_{n''i} x_{n'} x_i && 1 \leq n', n'' \leq n,\\ &\sum_i a_{n'i} x_{n'} x_i && 1 \leq n' \leq n,\\ &x_{n_1} \cdots x_{n_l} && 1 \leq n_i \leq n).\end{aligned}$$

Then

$$\min\{l-1, n-1\} \leq \dim \Lambda^A_{l,n}\text{-mod} \leq \dim D^b(\Lambda^A_{l,n}\text{-mod}) \leq \min\{l-1, n\},$$

and in particular

$$\operatorname{repdim} \Lambda^A_{l,n} \geq \min\{l+1, n+1\}.$$

7.4.1 Subexample. In Example 7.4 above, let $n = 3$, $l = 4$ and $A = \begin{pmatrix} st & & \\ & t & \\ & & 1 \end{pmatrix}$ with $s, t \in k \setminus \{0\}$. Then we find

$$\operatorname{repdim} \big(k\langle x, y, z\rangle / (x^2, y^2, z^2, xy + syx, xz + stzx, yz + tzy)\big) \geq 4.$$

7.5 Example (from Examples 6.2 and 7.3)**.** Let $A = (a_{ij})$ be an invertible $n \times n$-matrix over k. Let $\Sigma^A_{l,n}$ be the algebra

$$\begin{aligned}\Sigma^A_{l,n} = k\langle x_1, \ldots, x_n\rangle / (&\sum_i a_{n'i} x_{n''} x_i - \sum_i a_{n''i} x_{n'} x_i && 1 \leq n', n'' \leq n\\ &x_{n_1} \cdots x_{n_l} && 1 \leq n_i \leq n).\end{aligned}$$

Then

$$\min\{l-1, n-1\} \leq \dim \Sigma_{l,n}^{A}\text{-mod} \leq \dim D^b(\Sigma_{l,n}^{A}\text{-mod}) \leq l-1,$$

and in particular

$$\operatorname{repdim} \Sigma_{l,n}^{A} \geq \min\{l+1, n+1\}.$$

Appendix: Comparison with Iyama's upper bound for the representation dimension

The results presented here are based on the following theorem of Iyama. The application of his result to the examples was suggested by Iyama, who worked out in detail the upper bound for the representation dimension of the algebra considered by Krause and Kussin presented here as Example A.8 (private communication [12]).

Theorem 4 (Iyama [13, Theorem 2.2.2 and Theorem 2.5.1]). *Let Λ be a finite dimensional algebra. Let $M = M_0 \in \Lambda$-mod and $M_{i+1} = M_i \cdot J_{\operatorname{End}_\Lambda(M_i)}$ (remember that $J_{\operatorname{End}_\Lambda(M_i)}$ denotes the Jacobson radical of the algebra $\operatorname{End}_\Lambda(M_i)$). Assume $M_m = 0$. Then*

$$\operatorname{gld} \operatorname{End}_\Lambda(\bigoplus_i M_i) \leq m.$$

In particular, for $M = \Lambda \oplus \Lambda^*$,

$$\operatorname{repdim} \Lambda \leq m.$$

Here we only consider the case $M = \Lambda \oplus \Lambda^*$. We will show that the upper bound for the representation dimension provided by Iyama's theorem coincides with the lower bound we found for some of the algebras we considered.

The following corollary and its proof are a slight extension of a result shown by Iyama in a private letter [12].

A.1 Corollary. *Let Λ be a finite dimensional algebra with l simple modules $S_1, \ldots, S_l$ (up to isomorphism), such that $\operatorname{Ext}^1_\Lambda(S_v, S_w) = 0$ whenever $v \neq w-1$. Denote by I_v the injective module with socle S_v. Assume*

(1) $\operatorname{End}_\Lambda J^i_\Lambda P$ is semisimple for any i and any indecomposable projective module P.

(2) there is $1 \leq l_0 \leq l$ such that

(a) I_v is projective for all $v > l_0$, and

(b) *all composition factors of* $\operatorname{Soc} \Lambda$ *are among the simple modules corresponding to vertices* $l_0, \ldots, l$.

Then

$$\operatorname{repdim} \Lambda \leq \max\{\operatorname{LL} \Lambda, \max\{\operatorname{LL} I_v + 1 \mid I_v \textit{ not projective}\}\}.$$

A.2 Remark. The condition on the extensions between simple Λ-modules just means that the (valued) quiver of Λ is of the form

$$\underset{1}{\circ} \xrightarrow{(a_1,b_1)} \underset{2}{\circ} \xrightarrow{(a_2,b_2)} \cdots \xrightarrow{(a_{l-1},b_{l-1})} \underset{l}{\circ}$$

for arbitrary a_i, $b_i \in \mathbb{N}$.

Proof. Set $V = \{v \in \{1, \ldots, l\} \mid I_v \text{ not projective}\}$. We may assume that $l_0 \in V$. We apply Iyama's Theorem with $M_0 = \Lambda \oplus \bigoplus_{v \in V} I_v$. We will show that $M_i = J_\Lambda^i \oplus \bigoplus_{v \in V} I_v^i$, for submodules $I_v^i \subseteq I_v$ with $\operatorname{LL} I_v^i \leq \max_{v \in V} \operatorname{LL} I_v + 1 - i$.

Clearly the construction in Iyama's Theorem respects the direct sum decomposition of M_0.

We first look at morphisms to the submodules of the indecomposable injective non-projective modules.

Let $v \in V$ and $\varphi \in \operatorname{Hom}(J_\Lambda^i, I_v^i)$. Since I_v is injective φ extends to a map $\Lambda \longrightarrow I_v$ as indicated in the following diagram.

$$\begin{tikzcd} J_\Lambda^i \arrow[r, hook] \arrow[d, "\varphi"'] & \Lambda \arrow[d] \\ I_v^i \arrow[r, hook] & I_v \end{tikzcd}$$

Therefore the image of φ is contained in $J_\Lambda^i I_v$. Conversely let $\psi : \Lambda^n \twoheadrightarrow I_v$ be a projective cover. Since I_v is not projective this is in the radical of Λ-mod, so $I_v^1 = I_v$. Since embedding the radical is in the radical of Λ-mod so is the composition with the restriction of ψ to some radical power

$$(J_\Lambda^i)^n \overset{\psi}{\twoheadrightarrow} J_\Lambda^i I_v \hookrightarrow J_\Lambda^{i-1} I_v.$$

Therefore one can see, by induction over i, that $J_\Lambda^i \cdot \operatorname{Rad}_{\Lambda\text{-mod}}(J_\Lambda^i, I_v^i) = J_\Lambda^i I_v$ and $J_\Lambda^{i-1} I_v \subseteq I_v^i$. In particular we obtain $I_{l_0}^i = J_\Lambda^{i-1} I_{l_0}$, since there are no maps from the other I_v^i to $I_{l_0}^i$.

Now let $v, w \in V$ with $v \neq w$. Any map $\varphi : I_v^i \longrightarrow I_w^i$ has the socle of I_v^i in its kernel. Therefore the length of the image is at most the length of I_v^i

minus one. By induction on i one obtains $\operatorname{LL} I_v^i \leq \max_{w \in V} \operatorname{LL} I_w + 1 - i$ as claimed above.

Now we want to consider maps to the projective modules.

The composition of a projective cover with embedding of the radical $\Lambda^r \twoheadrightarrow J_\Lambda \hookrightarrow \Lambda$ restricts to maps $(J_\Lambda^i)^r \twoheadrightarrow J_\Lambda^{i+1} \hookrightarrow J_\Lambda^i$. These are in the radical of $\operatorname{End} J_\Lambda^i$ since the second map is in the radical of Λ-mod. Therefore, together with Assumption (1), we get $J_\Lambda^i \cdot J_{\operatorname{End}_\Lambda(J_\Lambda^i)} = J_\Lambda^{i+1}$.

For $v < l_0$ we have $\operatorname{Hom}_\Lambda(I_v^i, J_\Lambda^i) = 0$, since I_v^i and $\operatorname{Soc} J_\Lambda^i$ do not have any common composition factors. By looking at the composition factors, we can also see that any non-zero element of $\operatorname{Hom}_\Lambda(I_{l_0}^i, J_\Lambda^i)$ is a monomorphism. Now assume such a monomorphism exists. Remember that $I_{l_0}^i = J_\Lambda^{i-1} I_{l_0}$. Therefore the simple module corresponding to vertex $l_0 - (\operatorname{LL} I_{l_0} - 1) + (i-1) = l_0 - \operatorname{LL} I_{l_0} + i$ is a composition factor of $I_{l_0}^i$. So it also is a composition factor of J_Λ^i. Let w be a vertex such that it is a composition factor of $J_\Lambda^i P_w$, where P_w is the projective module corresponding to vertex w. Then $l_0 - \operatorname{LL} I_{l_0} + i \geq w + i$, so $l_0 - w \geq \operatorname{LL} I_{l_0}$. But if the simple module corresponding to l_0 is a composition factor of P_w then the simple module corresponding to w is a composition factor of I_{l_0}. Therefore $\operatorname{LL} I_{l_0} \geq l_0 - w + 1$. A contradiction. Therefore $\operatorname{Hom}_\Lambda(I_{l_0}^i, J_\Lambda^i) = 0$.

Putting everything together we find that M_i has the structure claimed above. In particular $M_{\max\{\operatorname{LL} \Lambda, \max\{\operatorname{LL} I_v + 1 \mid I_v \text{ not projective}\}\}} = 0$, so Iyama's Theorem provides the claim of the corollary. $\square$

A.3 Corollary. *Let Λ be a local finite dimensional algebra. Assume*

(1) $J_\Lambda^i \cdot J_{\operatorname{End}_\Lambda(J_\Lambda^i)} \subseteq J_\Lambda^{i+1}$ *for any i,*

(2) the socle and radical series coincide for Λ and for Λ^*, and

(3) any map $\operatorname{Soc}^3 \Lambda^* \longrightarrow \Lambda$ *has semisimple image.*

Then $\operatorname{repdim} \Lambda \leq \operatorname{LL} \Lambda + 1$.

Proof. We may assume that Λ is not self-injective (otherwise it is semisimple by Assumption (3)). We set $M_0 = \Lambda \oplus \Lambda^*$ and claim that $M_i = J_\Lambda^i \oplus J_\Lambda^{i-1} \Lambda^*$ for $i \geq 1$.

As in the proof of Corollary A.1 we can see that

$$J_\Lambda^i \Lambda^* \subseteq (J_\Lambda^i \oplus J_\Lambda^{i-1} \Lambda^*) \cdot \operatorname{Rad}_{\Lambda\text{-mod}}(J_\Lambda^i \oplus J_\Lambda^{i-1} \Lambda^*, J_\Lambda^{i-1} \Lambda^*) \subseteq \operatorname{Soc}^{\operatorname{LL} \Lambda - i} \Lambda^*.$$

Since both sides coincide by Assumption (2) we have equality.

The proof of $J_\Lambda^i \cdot J_{\operatorname{End}_\Lambda(J_\Lambda^i)} = J_\Lambda^{i+1} \Lambda$ is also identical to the proof of this equality in the case of Corollary A.1.

It remains to show that $J_\Lambda^{i-1}\Lambda^* \cdot \operatorname{Hom}_\Lambda(J_\Lambda^{i-1}\Lambda^*, J_\Lambda^i) \subseteq J_\Lambda^{i+1}\Lambda$. Unfortunately this will clearly fail for $i = \operatorname{LL}\Lambda - 1$. But in that case the image of any morphism to J_Λ^i has semisimple image (since the module is semisimple), and the simple module is a direct summand of M_{i+1} anyway, so it still agrees with our claim above. Now assume $i < \operatorname{LL}\Lambda - 1$. Let $\varphi \in \operatorname{Hom}_\Lambda(J_\Lambda^{i-1}\Lambda^*, J_\Lambda^i)$. We consider the following composition

$$\operatorname{Soc}^3 \Lambda^* \hookrightarrow J_\Lambda^{i-1}\Lambda^* \xrightarrow{\varphi} J_\Lambda^i \hookrightarrow \Lambda.$$

By Assumption (3) this composition has semisimple image, so φ factors through $J_\Lambda^{i-1}\Lambda^* \twoheadrightarrow J_\Lambda^{i-1}\Lambda^*/\operatorname{Soc}^2\Lambda^*$. Therefore the image has Loewy length at most $\operatorname{LL}\Lambda - (i-1) - 2 = \operatorname{LL}\Lambda - i - 1$, so it is contained in $\operatorname{Soc}^{\operatorname{LL}\Lambda - i - 1}\Lambda = J_\Lambda^{i+1}\Lambda$. □

A.4 Remark / Corollary. Auslander [2] has shown that the representation dimension of a self-injective algebra is bounded above by its Loewy length. This result also follows from Iyama's Theorem (similar to and easier than Corollary A.3).

Application to the examples: Let us start by checking Assumption (1) of Corollary A.3 for the exterior algebra.

A.5 Lemma. *Let* $\Lambda_N = k\langle x_1, \ldots, x_n\rangle / (x_{n''}x_{n'} + x_{n'}x_{n''}, x_{n'}^2)$ *be the exterior algebra. Let* $0 \leq i \leq j - 2 \leq n - 1$. *Then*

$$\operatorname{End}_{\Lambda_n} J^i\Lambda_n/J^j\Lambda_n = k \oplus \operatorname{Hom}_{\Lambda_n}(J^i\Lambda_n/J^j\Lambda_n, J^{i+1}\Lambda_n/J^j\Lambda_n)\iota$$

where ι *is the natural embedding.*

Proof. Since we are looking at a graded module the endomorphism ring is also graded. Therefore we only have to verify that all degree 0 endomorphisms are multiplication by a scalar.

If $i = 0$ or $j = n + 1$ this is true, since the endomorphisms induce endomorphisms of the simple head ($i = 0$) or simple socle ($j = n + 1$). Therefore we may exclude these cases in the next step.

Assume $1 \leq i$ and $j \leq n$. Let $\varphi : J^i\Lambda_n/J^j\Lambda_n \longrightarrow J^i\Lambda_n/J^j\Lambda_n$ be a degree 0 morphism. We want to show now that φ maps $x_{n'}J^{i-1}\Lambda_n/J^j\Lambda_n$ to itself for any n'. Let $p \in J^{i-1}\Lambda_n$ be an element of degree $i - 1$. Then

$$\begin{aligned} x_{n''}\varphi(x_{n'}p + J^j\Lambda_n) = \varphi(x_{n''}x_{n'}p + J^j\Lambda_n) &= -x_{n'}\varphi(x_{n''}p + J^j\Lambda_n) \\ &\in x_{n'} \cdot J^i\Lambda_n/J^j\Lambda_n. \end{aligned}$$

Let

$$\varphi(x_{n'}p + J^j\Lambda_n) = \sum_{n_1<n_2<\cdots<n_i} \alpha_{n_1,\ldots,n_i} x_{n_1}\cdots x_{n_i}.$$

Then

$$x_{n''}\varphi(x_{n'}p + J^j\Lambda_n) = \sum_{\substack{n_1<n_2<\cdots<n_i \\ n_1\neq n'',\cdots,n_i\neq n''}} \alpha_{n_1,\ldots,n_i} x_{n''}x_{n_1}\cdots x_{n_i}.$$

Therefore each monomial with a nonzero coefficient in $\varphi(x_{n'}p+J^j\Lambda_n)$ contains at least one of $x_{n'}$ and $x_{n''}$. Since this works for any $n'' \neq n'$ each such monomial contains $x_{n'}$ or all other $x_{n''}$. The latter case cannot occur, since $i < n-1$, so φ maps $x_{n'}J^{i-1}\Lambda_n/J^j\Lambda_n$ to itself as claimed above.

Now we show by induction on i and simultaneously for all $n > i$ that any degree 0 morphism $\varphi : J^i\Lambda_n/J^j\Lambda_n \longrightarrow J^i\Lambda_n/J^j\Lambda_n$ is multiplication by a scalar.

For $i = 0$ this is clear, so assume $i > 0$. Then we know that φ maps $x_{n'} \cdot J^{i-1}\Lambda_n/J^j\Lambda_n$ to itself. Now $x_{n'} \cdot J^{i-1}\Lambda_n/J^j\Lambda_n = J^{i-1}\Lambda_{n-1}/J^{j-1}\Lambda_{n-1}$, where the Λ_{n-1} is to be interpreted as the exterior algebra on the vector space generated by $x_{n''}$ with $n'' \neq n'$. Now inductively $\varphi|_{x_{n'}\cdot J^{i-1}\Lambda_n/J^j\Lambda_n}$ is multiplication by some scalar $\alpha_{n'}$, and the $\alpha_{n'}$ all coincide since the $x_{n'} \cdot J^{i-1}\Lambda_n/J^j\Lambda_n$ have pairwise non-trivial intersection.

Therefore our morphism is multiplication by a scalar, and the claim of the lemma follows. □

A.6 Example. Let $\Lambda_{l,n}$ be the family of algebras from Example 6.1. That is $\Lambda_{l,n} = k\langle x_1,\ldots,x_n\rangle / (x_{n''}x_{n'} + x_{n'}x_{n''}, x_{n'}^2, x_{n_1}\cdots x_{n_l})$. Assume $l \neq n$. Then

$$\operatorname{repdim}\Lambda_{l,n} \leq \min\{l+1, n+1\}.$$

Together with Example 6.1 this implies that we have equality here and $\dim \Lambda_{l,n}\text{-mod} = \min\{l-1, n-1\}$.

Proof. In case $l \geq n+1$ the algebra is self-injective and the claim follows from Remark A.4. Otherwise we would like to apply Corollary A.3. We have verified Assumption (1) in Lemma A.5, and Assumption (2) is obvious. To verify Assumption (3) let $\varphi : \operatorname{Soc}^3\Lambda_{l,n}^* \longrightarrow \Lambda_{l,n}$. The monomials of the form $x_{n_1}\cdots x_{n_i}$ with $n_1 < \cdots < n_i$ and $i < l$ form a basis of $\Lambda_{l,n}$. We consider the dual basis of $\Lambda_{l,n}^*$. Then

$$\operatorname{Soc}^3\Lambda_{l,n}^* = \bigoplus_{\substack{m \text{ such a} \\ \text{monomial of} \\ \text{degree } \leq 2}} km^*.$$

Now note that for $n' < n''$ we have $x_r(x_{n'}x_{n''})^* = 0$ for all $r \notin \{n', n''\}$. Therefore all these x_r have to operate as zero on $\varphi((x_{n'}x_{n''})^*)$. It follows that $\varphi((x_{n'}x_{n''})^*) \in J^{n-2}\Lambda_{l,n} + \operatorname{Soc}\Lambda_{l,n}$. Since $\operatorname{Soc}\Lambda_{l,n} = J^{l-1}\Lambda_{l,n}$, Claim (3) follows for $n-2 \geq l-1$, that is $l \leq n-1$. $\square$

Unfortunately, Iyama's Theorem does not give us the desired bound for $l = n$.

A.7 Example. Let $\widetilde{\Lambda}_{l,n}$ be the family of algebras from Example 7.2, that is $\widetilde{\Lambda}_{l,n} = kQ/I$ with

$$Q = \overset{1}{\circ} \overset{x_1}{\underset{x_n}{\overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}}}} \overset{2}{\circ} \overset{x_1}{\underset{x_n}{\overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}}}} \overset{3}{\circ} \quad \cdots \quad \overset{l-1}{\circ} \overset{x_1}{\underset{x_n}{\overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}}}} \overset{l}{\circ} \text{ and}$$
$$I = (x_{n''}x_{n'} + x_{n'}x_{n''}, x_{n'}^2).$$

Then

$$\operatorname{repdim}\widetilde{\Lambda}_{l,n} \leq \min\{l+1, n+1\}.$$

Together with Example 7.2 this implies that we have equality here and $\dim\widetilde{\Lambda}_{l,n}\text{-mod} = \min\{l-1, n-1\}$.

Proof. This time we want to apply Corollary A.1. One can see that Assumption (1) is satisfied as in the proof of Lemma A.5, except that we do not have to restrict to degree 0 morphisms (since there are no morphisms of non-zero degree). If we choose $l_0 = \min\{l, n\}$ then Assumption (2) holds. $\square$

A.8 Example (shown by Iyama)**.** Let $\widetilde{\Sigma}_{l,n}$ be the family of algebras from Example 7.3, that is $\widetilde{\Sigma}_{l,n} = kQ/I$ with

$$Q = \overset{1}{\circ} \overset{x_1}{\underset{x_n}{\overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}}}} \overset{2}{\circ} \overset{x_1}{\underset{x_n}{\overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}}}} \overset{3}{\circ} \quad \cdots \quad \overset{l-1}{\circ} \overset{x_1}{\underset{x_n}{\overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}}}} \overset{L}{\circ} \text{ and}$$
$$I = (x_{n''}x_{n'} - x_{n'}x_{n''}).$$

Then

$$\operatorname{repdim}\widetilde{\Sigma}_{l,n} \leq l+1.$$

In particular for $n \geq l$ we have equality, by Example 7.3.

Proof. We apply Corollary A.1. Assumption (1) can again be seen similarly to the proof of Lemma A.5, by combining the changes sketched in the proofs of Examples A.7 and A.9. Assumption (2) clearly holds for $l_0 = l$. $\square$

A.9 Example. Let $\Sigma_{l,n}$ be the family of algebras from Example 6.2. That is $\Sigma_{l,n} = k[x_1, \dots, x_n]/(x_1, \dots, x_n)^l$. Then

$$\operatorname{repdim} \Sigma_{l,n} \leq l + 1.$$

In particular for $n \geq l$ we have equality by Example 6.2.

Proof. We want to apply Corollary A.3. We can see that Assumption (1) holds in a similar way to the proof of Lemma A.5. The differences are that we cannot and don't have to exclude the case $j = l - 1$, and that $x_n \cdot J^{i-1}\Sigma_{l,n}/J^j\Sigma_{l,n} = J^{i-1}\Sigma_{l,n}/J^{j-1}\Sigma_{l,n}$, so we do not need to look at different n simultaneously. Assumption (2) is obvious and Assumption (3) can be seen as in Example A.6. $\square$

A.10 Remark. The case $n = 1$ suggests that in the last two examples the correct number for the representation dimension could be $\min\{l + 1, n + 1\}$.

References

[1] Maurice Auslander, *Representation dimension of Artin algebras*, Queen Mary College Mathematics Notes (1971), republished in [3].

[2] ______, *Representation theory of Artin algebras. I, II*, Comm. Algebra **1** (1974), 177–268; ibid. 1 (1974), 269–310.

[3] ______, *Selected works of Maurice Auslander. Part 1*, American Mathematical Society, Providence, RI, 1999, Edited and with a foreword by Idun Reiten, Sverre O. Smalø, and Øyvind Solberg.

[4] L. L. Avramov, R.-O. Buchweitz, S. Iyengar, and C. Miller, *Homology of perfect complexes*, preprint, arXiv:math.AC/0609008.

[5] L. L. Avramov and S. Iyengar, *The dimension of the stable derived category of a complete intersection local ring*, in preparation.

[6] A. A. Beĭlinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171.

[7] A. Bondal and M. van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Mosc. Math. J. **3** (2003), no. 1, 1–36, 258.

[8] J. Daniel Christensen, *Ideals in triangulated categories: phantoms, ghosts and skeleta*, Adv. Math. **136** (1998), no. 2, 284–339.

[9] Charles W. Curtis and Irving Reiner, *Representation theory of finite groups and associative algebras*, Pure and Applied Mathematics, Vol. XI, Interscience Publishers, a division of John Wiley & Sons, New York-London, 1962.

[10] Karin Erdmann, Thorsten Holm, Osamu Iyama, and Jan Schröer, *Radical embeddings and representation dimension*, Adv. Math. **185** (2004), no. 1, 159–177.

[11] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.

[12] Osamu Iyama, private communication.

[13] ———, *Rejective subcategories of artin algebras and orders*, preprint, arXiv:math.RT/0311281.

[14] ———, *Finiteness of representation dimension*, Proc. Amer. Math. Soc. **131** (2003), no. 4, 1011–1014.

[15] ———, *Representation dimension and Solomon zeta function*, Representations of finite dimensional algebras and related topics in Lie theory and geometry, Fields Inst. Commun., vol. 40, Amer. Math. Soc., Providence, RI, 2004, pp. 45–64.

[16] Bernhard Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) **27** (1994), no. 1, 63–102.

[17] Henning Krause and Dirk Kussin, *Rouquier's theorem on representation dimension*, Trends in representation theory of algebras and related topics, Contemp. Math., vol. 406, Amer. Math. Soc., Providence, RI, 2006, pp. 95–103.

[18] Hideyuki Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid.

[19] Steffen Oppermann, *A lower bound for the representation dimension of kC_p^n*, Math. Z. **256** (2007), no. 3, 481–490.

[20] Jeremy Rickard, *Derived categories and stable equivalence*, J. Pure Appl. Algebra **61** (1989), no. 3, 303–317.

[21] Raphaël Rouquier, *Representation dimension of exterior algebras*, Invent. Math. **165** (2006), no. 2, 357–367.

[22] ______, *Dimensions of triangulated categories*, J. K-theory **1** (2008), no. 2, 193–256.